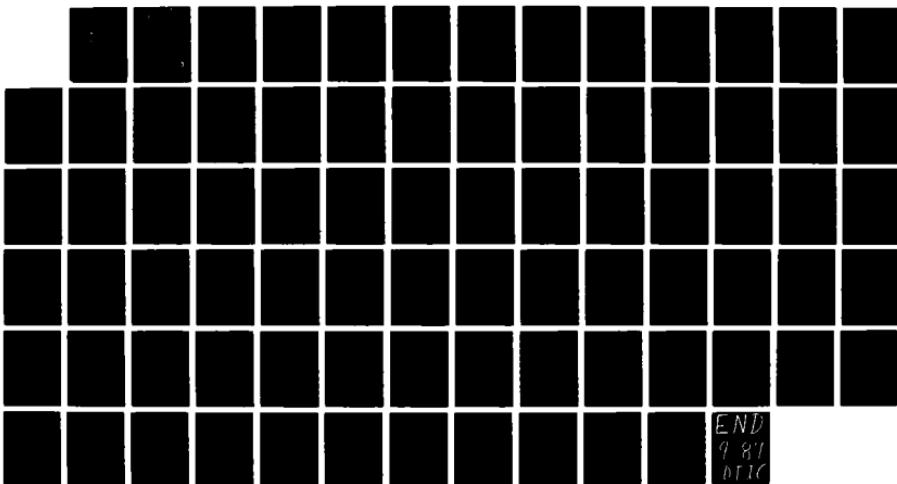


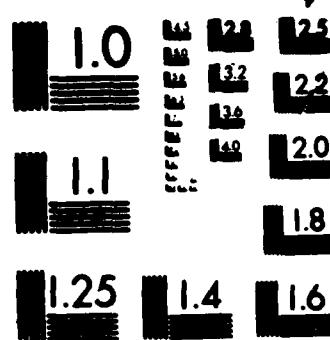
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EIGENFUNCTIONS AT A SINGULAR POINT FOR TRANSVERSELY
ISOTROPIC COMPOSITES WITH APPLICATIONS TO STRESS
ANALYSIS OF A BROKEN FIBER

January 1987

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ABSTRACT

When a transversely isotropic elastic body that contains a notch or a crack is under an axisymmetric deformation, it is shown that the eigenfunction solution near the singular point is in the form of a power series, $\rho^{\delta} F_0(\psi, \delta), \rho^{\delta+1} F_1(\psi, \delta), \rho^{\delta+2} F_2(\psi, \delta) \dots$ in which (ρ, ψ) is the polar coordinate with origin at the singular point and δ is the eigenvalue, or the order of singularity. A difficulty arises when δ as well as $\delta+k$ where k is a positive integer is also an eigenvalue. In this case the higher order terms of the series solution may not exist. A modified solution is required and is presented here. The modified solution has the new terms $\rho^{\delta+k} (\ln \rho) F_0(\psi, \delta), \rho^{\delta+k+1} (\ln \rho) F_1(\psi, \delta) \dots$ As an application, we consider the stresses near a broken fiber in a composite which is under an axisymmetric deformation. The interface between the broken fiber and the matrix also suffers a delamination. This creates stress singularities at several points some of which require the modified eigenfunctions presented here.

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FOREWORD

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Chapter I

INTRODUCTION

Even though light weight, high strength composites have been widely used in the industries, rigorous analysis of stress distribution in a composite which contains delaminations and/or broken fibers are still lacking. The difficulties are due to the presence of stress singularities at the singular points such as the interface crack tip and the edges of the broken fibers. Accurate predictions of the stresses near these singular points are important not only for studies of fracture behavior of materials but also for studies of general stress analysis. In finding stress distribution in an entire specimen numerically by a finite element scheme, one may use regular finite elements everywhere except at the singular points. At these singular points, special elements are used in which the singular nature of the stress is given by an analytical expression.

The problem of finding the stress singularities at the apex of an isotropic elastic wedge or notch was first considered by Knein (1926) and Williams (1952) in which they assume that the stress distribution under a plane-stress or plane-strain deformation can be expressed in terms of a series of eigenfunctions of the form $\rho^\delta f(\psi, \delta)$ where ρ is the radial distance from the apex and the f is a function of the polar angle ψ and the eigenvalue δ . For given wedge angle and homogeneous boundary conditions at the sides of the wedge, there are in general infinitely

many eigenvalues δ and the associated eigenfunctions $\rho^\delta f(\psi, \delta)$. Particularly important in applications is when one or more of the δ 's is negative and the stress is singular at the apex. It was shown that for a two dimensional body under an external loading the negative δ appears when the wedge angle is larger than π and $\delta = -1/2$ is a double root when the wedge angle is 2π (i.e. the case of crack). The technique is applied to a crack along (Williams 1959, England 1965), and normal (Zak and Williams 1963, Cook and Erdogan 1972) to the interface and to other geometries of isotropic composites (Bogy 1970, 1971, Bogy and Wang 1971, Erdogan and Gupta 1972, Delale et al 1984). A systematic derivation of the equation for finding the singularity δ was given by Dempsey and Sinclair (1981).

Investigation of associated problems for anisotropic materials was started by Sih et al (1965) and has become active only in the last decade (for example, Bogy 1972, Kuo and Bogy 1974, Delale and Erdogan 1979, Sih and Chen 1981, Hoenig 1982). However, these studies are limited to two-dimensional singular points. There are singular points which are three-dimensional. Three-dimensional singularity analysis for isotropic materials was first performed analytically by Benthem (1977, 1980) and Kawai et al (1977) and numerically by Bazant (1974) and Bazant and Estenssoro (1977, 1980). Extension to anisotropic materials and composites was considered recently by Somaratna and Ting (1986A, B). In Somaratna and Ting (1986B) finite element schemes are employed to determine the order of singularity at a three-dimensional singular point of any geometry. In the other paper of Somaratna and Ting (1986A) the order of singularity is determined analytically for the special case of tran-

versely isotropic materials under an axisymmetric deformation. The singular point is assumed to locate on the axis of symmetry. In present study, we consider the case in which the singular point is not on the axis of symmetry.

Fig.1 shows the cross section of an axisymmetric body under an axisymmetric deformation. The material is assumed to be transversely isotropic with the z-axis being the axis of symmetry. We are interested in the stresses near the singular point R. The associated problem for isotropic materials was investigated by Delale and Erdogan (1981) and Delale et al (1984). However, their objectives are different from ours and hence their series solution is different from the one presented here.

A series solution for the problem is developed in Chapter II. After presenting the basic equations for transversely isotropic materials under an axisymmetric deformation in Section 2.1, the general solution in the form of a power series in ρ is presented in Section 2.2. Application of the stress-free conditions at the sides of the wedge leads to equations for the eigenvalue δ and the coefficients in the power series. This is presented in Section 2.3. It is seen that the eigenfunction associated with an eigenvalue δ no longer contains a single term $\rho^\delta f(\psi, \delta)$. It also has the terms $\rho^{\delta+1} f_1(\psi, \delta)$, $\rho^{\delta+2} f_2(\psi, \delta)$... Therefore, the inclusion of the second and higher-order terms in the special element is not simply the inclusion of the eigenfunctions associated with the subsequent smallest eigenvalue δ . A similar situation occurs in wedge with curved sides under a 2-dimensional deformation (Ting 1985). The derivations presented in Section 2.2 and 2.3 are for

the general case in which the two eigenvalues p_1 , p_2 of the elasticity constants are distinct. The degenerate case in which $p_1 = p_2$ (of which the isotropic material is a special case) is discussed in Section 2.4. For 2-dimensional deformations, the displacement of the singular point R (Fig.1) can be ignored for the singularity analysis. For axisymmetric deformations, one cannot ignore the displacement of the singular point R in the r -direction. A particular solution associated with the displacement of the singular point is presented in Section 2.5. A difficulty arises when δ as well as $\delta+k$, where k is a positive integer, is an eigenvalue. In this case the higher order terms of the series solution of the eigenfunction cannot always be determined. A modified solution is required and is presented in Chapter III. We can see in Section 3.1 and Appendix B that the modified eigenfunction solution has the new terms $\rho^{\delta+k}(\ln\rho)F_1(\psi, \delta)$, $\rho^{\delta+k+1}(\ln\rho)F_2(\psi, \delta)$... Application of the stress-free boundary condition is presented in Section 3.2. The solutions of Chapter II are then applied to composite materials in Chapter IV. The equations for general transversely isotropic materials and degenerate materials are presented in Section 4.2 and 4.3, respectively, and numerical examples are given in Section 4.4. For the singular point which is the tip of an interface crack the displacement was found to be oscillatory. This implies that the two crack surfaces inter-penetrate each other. To avoid the unrealistic phenomenon a contact zone near the crack tip is introduced in Chapter V. In Chapter VI the formulas for singularities at an interface crack with contact zone are derived by using the Stroh formalism. This alternate approach offers an analytical solution for the singularity δ which agrees with the numerical results obtained in Chapter V.

Chapter II

EIGENFUNCTION FOR AXI-SYMMETRIC DEFORMATIONS

2.1 MATHEMATICAL FORMULATION

Let (r, θ, z) be a cylindrical coordinate system with the z -axis as the axis of material symmetry and let (u_r, u_θ, u_z) be the corresponding displacement components. We assume that the deformation is axisymmetric and $u_\theta = 0$ so that u_r and u_z are functions of r and z only. Introducing the displacement potential $\phi(r, z)$ which gives u_r and u_z by (Elliott 1948, Green and Zerna 1954 and Kassir and Sih 1975)

$$u_r = \frac{\partial \phi}{\partial r}, \quad u_z = m \frac{\partial \phi}{\partial z}, \quad (2.1)$$

where m is a constant to be determined, the stresses are obtained as

$$\sigma_r = c_{11} \frac{\partial^2 \phi}{\partial r^2} + c_{12} \frac{\partial \phi}{r \partial r} + c_{13} m \frac{\partial^2 \phi}{\partial z^2}, \quad (2.2)$$

$$\sigma_\theta = c_{12} \frac{\partial^2 \phi}{\partial r^2} + c_{11} \frac{\partial \phi}{r \partial r} + c_{13} m \frac{\partial^2 \phi}{\partial z^2},$$

$$\sigma_z = c_{13} \frac{\partial^2 \phi}{\partial r^2} + c_{13} \frac{\partial \phi}{r \partial r} + c_{33} m \frac{\partial^2 \phi}{\partial z^2},$$

$$\sigma_{rz} = c_{44} (1+m) \frac{\partial^2 \phi}{\partial r \partial z},$$

in which c_{ij} are the elasticity constants for the transversely isotropic material. The equation of equilibrium are satisfied if

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{\partial \phi}{r \partial r} - \frac{1}{p^2} \frac{\partial^2 \phi}{\partial z^2} = 0, \quad (2.3)$$

where

$$p^2 = \frac{-c_{11}}{mc_{13} + (1+m)c_{44}} = \frac{c_{13} + (1+m)c_{44}}{-mc_{33}}, \quad (2.4a)$$

or, equivalently,

$$-m = \frac{c_{11} + c_{44} p^2}{(c_{13} + c_{44}) p^2} = \frac{c_{13} + c_{44}}{c_{44} + c_{33} p^2}. \quad (2.4b)$$

The second equality of (2.4a) and (2.4b) respectively yield

$$m^2 - 2 \left[\frac{c_{11}c_{33} - c_{13}^2}{2c_{44}(c_{13} + c_{44})} - 1 \right] m + 1 = 0, \quad (2.5a)$$

$$p^4 + 2 \left[\frac{c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44}}{2c_{33}c_{44}} \right] p^2 + \frac{c_{11}}{c_{33}} = 0. \quad (2.5b)$$

It can be shown (Eshelby et al 1953) that p cannot be real if the strain energy is positive definite. Therefore we have two pairs of complex conjugates for p which will be denoted by p_1, \bar{p}_1, p_2 and \bar{p}_2 where an overbar indicates the complex conjugate. The associated values of m are denoted by $m_1, \bar{m}_1, m_2, \bar{m}_2$ respectively. From (2.5a) we note that

$$m_1 m_2 = 1. \quad (2.5c)$$

Since (2.5b) is a quadratic equation in p^2 with real coefficients, if p_1 is purely imaginary so is p_2 . Then m_1, m_2 are real and $\bar{m}_1 = m_1, \bar{m}_2 = m_2$. If p_1 and p_2 are not purely imaginary we may choose

$$p_1 = u + iv = -\bar{p}_2, \quad p_2 = -u + iv = -\bar{p}_1, \quad (2.6)$$

where u , v are real. In this case m_1 and m_2 are complex and $m_1 = \bar{m}_2$. In view of the fact that the equations are linear, the general solution for ϕ is obtained by superposing ϕ 's associated with p_1 , \bar{p}_1 , p_2 , \bar{p}_2 . We will assume that $p_1 \neq p_2$. The degenerate case in which $p_1 = p_2$ will be discussed in Section 2.4.

2.2 EIGENFUNCTIONS FOR SMALL ρ

Let $(r, z) = (a, 0)$ be a singular point which may be the apex of a wedge, notch, crack, or the tip of an interface crack. We now consider the case in which $a \neq 0$. The case in which $a = 0$ has been studied by Somaratna and Ting (1986A). Using the singular point as the origin, we define

$$x = r-a = \rho \cos\psi, \quad z = \rho \sin\psi. \quad (2.7)$$

To find the eigenfunction for ϕ that is valid for small ρ , we rewrite equation (2.3) as

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2 \phi}{\partial z^2} = - \frac{1}{a+x} \frac{\partial \phi}{\partial x} = - \frac{1}{a} \frac{\partial \phi}{\partial x} \sum_{s=0}^{\infty} \left(\frac{-x}{a} \right)^s. \quad (2.8)$$

Let

$$\phi = \phi^{(0)} - \frac{1}{a} \phi^{(1)} + \frac{1}{a^2} \phi^{(2)} - \dots = \sum_{k=0}^{\infty} \left(\frac{-1}{a} \right)^k \phi^{(k)}, \quad (2.9a)$$

$$\phi^{(k)} = \sum_{t=0}^k A_t^{(k)} x^t z^{\delta+k-t+2}, \quad (2.9b)$$

$$z = x + pz, \quad (2.10)$$

where δ is the eigenvalue and $A_t^{(k)}$ are constants to be determined. Using equation (2.7) we have

$$x^t z^{\delta+k-t+2} = \rho^{\delta+k+2} (\cos \psi)^t \xi^{\delta+k-t+2}, \quad (2.11)$$

$$\xi = \cos \psi + p \sin \psi. \quad (2.12)$$

Therefore $\phi^{(k)}$ is of order $\rho^{\delta+k+2}$. By substituting equations (2.9) into (2.8) and equating the coefficients of $x^t z^{\delta+k-t+2}$, it can be shown that (see Appendix A)

$$A_k^{(k)} = \frac{2k-1}{2k} A_{k-1}^{(k-1)}, \quad (k > 0), \quad (2.13a)$$

$$A_t^{(k)} = \frac{2t-1}{2t} A_{t-1}^{(k-1)} - \frac{1}{2(\delta+k-t+2)} [t A_t^{(k-1)} - (t+1) A_{t+1}^{(k)}]. \quad (2.13b)$$

(t = k-1, ..., 1)

Hence the only unknowns are $A_0^{(k)}$ ($k=0, 1, 2, \dots$) and δ which will be determined from the boundary conditions.

We will let the solution given by equations (2.9)-(2.13) apply to $p = p_1$. For $p = p_2$, \bar{p}_1 , and \bar{p}_2 , we will use the same expressions except that $A_t^{(k)}$ is replaced by $B_t^{(k)}$, $C_t^{(k)}$ and $D_t^{(k)}$, respectively. Thus the general solution for $\phi^{(k)}$ is

$$\phi^{(k)} = \sum_{t=0}^k \{ A_t^{(k)} x^t z_1^{\delta+k-t+2} + B_t^{(k)} x^t z_2^{\delta+k-t+2} + C_t^{(k)} x^t z_1^{-\delta+k-t+2} + D_t^{(k)} x^t z_2^{-\delta+k-t+2} \}, \quad (2.14a)$$

$$z_s = x + p_s z \quad (s=1, 2). \quad (2.14b)$$

Substituting (2.14) into (2.1), we obtain

$$\begin{aligned}
 u_r^{(k)} = & \left\{ \sum_{t=0}^k A_t^{(k)} (\delta+k-t+2) + \sum_{t=0}^{k-1} A_{t+1}^{(k)} (t+1) \right\} x^t z_1^{\delta+k-t+1} \\
 & + \left\{ \sum_{t=0}^k B_t^{(k)} (\delta+k-t+2) + \sum_{t=0}^{k-1} B_{t+1}^{(k)} (t+1) \right\} x^t z_2^{\delta+k-t+1} \\
 & + \left\{ \sum_{t=0}^k C_t^{(k)} (\delta+k-t+2) + \sum_{t=0}^{k-1} C_{t+1}^{(k)} (t+1) \right\} x^t z_1^{\delta+k-t+1} \\
 & + \left\{ \sum_{t=0}^k D_t^{(k)} (\delta+k-t+2) + \sum_{t=0}^{k-1} D_{t+1}^{(k)} (t+1) \right\} x^t z_2^{\delta+k-t+1} .
 \end{aligned} \tag{2.15}$$

To simplify the expression, we will let

$$A_t^{(k)} = B_t^{(k)} = C_t^{(k)} = D_t^{(k)} = 0, \quad \text{if } t > k, \tag{2.16}$$

and rewrite equation (2.15) as

$$u_r^{(k)} = \sum_{t=0}^k [A_t^{(k)} (\delta+k-t+2) + A_{t+1}^{(k)} (t+1)] x^t z_1^{\delta+k-t+1} + \dots \tag{2.17a}$$

where the dots stand for the similar expressions associated with $p = p_2$, \bar{p}_1 , and \bar{p}_2 . Likewise, we have

$$u_z^{(k)} = \sum_{t=0}^k A_t^{(k)} \mathbf{m}, p_1, (\delta+k-t+2) x^t z_1^{\delta+k-t+1} + \dots \tag{2.17b}$$

In substituting equation (2.14) into (2.2) for the stresses, we first replace the terms $\partial\phi/\partial r$ in (2.2) by $\partial^2\phi/\partial x^2$ and $\partial^2\phi/\partial z^2$ using equation (2.3). We then have

$$\begin{aligned}
 \sigma_r^{(k)} = & \sum_{t=0}^k [-A_t^{(k)} c_{44} (1+m_1) p_1^2 (\delta+k-t+2) (\delta+k-t+1) \\
 & + A_{t+1}^{(k)} 2(c_{11} - c_{22}) (t+1) (\delta+k-t+1) \\
 & + A_{t+2}^{(k)} (c_{11} - c_{22}) (t+2) (t+1)] x^t z_1^{\delta+k-t} + \dots
 \end{aligned} \tag{2.18a}$$

$$\begin{aligned}\sigma_{\theta}^{(k)} = & \sum_{t=0}^k [A_t^{(k)} (c_{1,2} + c_{1,3}m, p_1^2) (\delta+k-t+2) (\delta+k-t+1) \\ & - A_{t+1}^{(k)} 2(c_{1,1} - c_{1,2})(t+1) (\delta+k-t+1) \\ & - A_{t+2}^{(k)} (t+2)(t+1)] x^t z_1^{\delta+k-t} + \dots\end{aligned}\quad (2.18b)$$

$$\sigma_z^{(k)} = \sum_{t=0}^k -A_t^{(k)} c_{4,4} (1+m_1) (\delta+k-t+2) (\delta+k-t+1) x^t z_1^{\delta+k-t} + \dots \quad (2.18c)$$

$$\sigma_{rz}^{(k)} = \sum_{t=0}^k [A_t^{(k)} (\delta+k-t+2) + A_{t+1}^{(k)} (t+1)] c_{4,4} (1+m_1) p_1 (\delta+k-t+1) x^t z_1^{\delta+k-t} + \dots \quad (2.18d)$$

In (2.18a) and (2.18c), the following identities which are obtained from equation (2.4a) have been used:

$$c_{1,1} + c_{1,3}mp^2 = -c_{4,4}(1+m)p^2, \quad (2.19a)$$

$$c_{1,3} + c_{3,3}mp^2 = -c_{4,4}(1+m). \quad (2.19b)$$

2.3 DETERMINATION OF δ AND $A_0^{(k)} \dots D_0^{(k)}$

The problem reduces to the determination of δ and $A_0^{(k)} \dots D_0^{(k)}$.

The stress-free boundary conditions at $\psi = \alpha$ and α' are

$$\sigma_r \sin \psi - \sigma_{rz} \cos \psi = 0, \quad (2.20a)$$

$$\sigma_{rz} \sin \psi - \sigma_z \cos \psi = 0.$$

Written in matrix notation, we have

$$\underline{\underline{N}}(\psi) \underline{\underline{\sigma}} = \underline{0}, \quad (\psi = \alpha, \alpha'), \quad (2.20b)$$

where

$$\underline{\underline{N}}(\psi) = \begin{bmatrix} \sin \psi & -\cos \psi & 0 \\ 0 & \sin \psi & -\cos \psi \end{bmatrix}, \quad (2.21)$$

$$\underline{\sigma} = \sum_{k=0}^{\infty} \underline{\sigma}^{(k)}, \quad \underline{\sigma}^{(k)} = \begin{bmatrix} \sigma_r^{(k)} \\ \sigma_{rz}^{(k)} \\ \sigma_z^{(k)} \end{bmatrix}, \quad (t = 0, 1, 2 \dots). \quad (2.22)$$

From equations (2.18) we may write $\underline{\sigma}^{(k)}$ as, using equation (2.11),

$$\underline{\sigma}^{(k)} = \rho^{\delta+k} \sum_{t=0}^k \{ \underline{S}_t(\psi, \delta+k) \underline{g}_t^{(k)} + \underline{I}_t(\psi, \delta+k) \underline{g}_{t+1}^{(k)} + \underline{U}_t(\psi, \delta+k) \underline{g}_{t+2}^{(k)} \}, \quad (2.23)$$

in which

$$\underline{g}_t^{(k)} = \begin{bmatrix} A_t^{(k)} \\ B_t^{(k)} \\ C_t^{(k)} \\ D_t^{(k)} \end{bmatrix}, \quad (2.24)$$

$$\underline{S}_t(\psi, \delta) = c_{44} (\delta-t+2)(\delta-t+1) \begin{bmatrix} -(1+m_1)p_1^2 & * & * & * \\ (1+m_1)p_1 & * & * & * \\ -(1+m_1) & * & * & * \end{bmatrix} \underline{Q}_t(\psi, \delta), \quad (2.25a)$$

$$\underline{I}_t(\psi, \delta) = (t+1)(\delta-t+1) \begin{bmatrix} 2(c_{11} - c_{12}) & * & * & * \\ c_{44}(1+m_1)p_1 & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{Q}_t(\psi, \delta), \quad (2.25b)$$

$$\underline{U}_t(\psi, \delta) = (t+2)(t+1)(c_{11} - c_{12}) \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{Q}_t(\psi, \delta). \quad (2.25c)$$

In equations (2.25), the second column of the matrix is obtained from the first column by replacing p_1 by p_2 (and of course m_1 by m_2). The third and fourth columns are, respectively, the complex conjugate of the first and second columns. $\underline{Q}_t(\psi, \delta)$ is a diagonal matrix given by

$$\tilde{Q}_t(\psi, \delta) = (\cos\psi)^t \text{diag} [\tilde{\xi}_1^{\delta-t}, \tilde{\xi}_2^{\delta-t}, (\tilde{\xi}_3)^{\delta-t}, (\tilde{\xi}_4)^{\delta-t}] . \quad (2.26)$$

$\tilde{q}_t^{(k)}$ of equation (2.24) is related to $\tilde{q}_t^{(k-1)}$ by using equations (2.13). Notice that equation (2.13a) can be regarded as a special case of equation (2.13b) if we use equation (2.16) and allow $t = k$ in (2.13b). Thus we have, for $k \geq 1$,

$$\tilde{q}_t^{(k)} = \frac{2t-1}{2t} \tilde{q}_{t-1}^{(k-1)} + \frac{1}{2(\delta+k-t+2)} [t\tilde{q}_t^{(k-1)} - (t+1)\tilde{q}_{t+1}^{(k)}], \quad (t = k, k-1, \dots, 1) \quad (2.27a)$$

$$\tilde{q}_t^{(k)} = \tilde{0}, \quad \text{if } t > k . \quad (2.27b)$$

As in (2.13) the only unknowns are δ and $\tilde{q}_0^{(k)}$ ($k = 0, 1, 2, \dots$).

Before we substitute (2.21)–(2.23) into (2.20b), we rewrite (2.23) as, making use of (2.27b)

$$\begin{aligned} \tilde{q}_0^{(k)} &= \rho^{\delta+k} \{ \tilde{S}_0(\psi, \delta+k) \tilde{q}_0^{(k)} \\ &+ \sum_{t=1}^k [\tilde{S}_t(\psi, \delta+k) + \tilde{I}_{t-1}(\psi, \delta+k) + \tilde{U}_{t-2}(\psi, \delta+k)] \tilde{q}_t^{(k)} \}, \end{aligned} \quad (2.28a)$$

where we have defined

$$\tilde{U}_t(\psi, \delta+k) = \tilde{0}, \quad \text{if } t < 0 . \quad (2.28b)$$

Now substitution of (2.21), (2.22), and (2.28a) into (2.20b) for $\psi = \alpha$ and α' yields the following equations for $\tilde{q}_0^{(k)}$:

$$\tilde{R}(\delta) \tilde{q}_0^{(0)} = \tilde{0}, \quad (2.29a)$$

$$\tilde{R}(\delta+k) \tilde{q}_0^{(k)} = - \sum_{t=1}^k \tilde{W}_t(\delta+k) \tilde{q}_t^{(k)} \quad (k \geq 1) , \quad (2.29b)$$

in which

$$\underline{K}(\delta) = \begin{bmatrix} \underline{N}(\alpha) \underline{S}_0(\alpha, \delta) \\ \underline{N}(\alpha') \underline{S}_0(\alpha', \delta) \end{bmatrix}, \quad (2.30a)$$

$$\underline{W}_t(\delta) = \begin{bmatrix} \underline{N}(\alpha) [\underline{S}_t(\alpha, \delta) + \underline{T}_{t-1}(\alpha, \delta) + \underline{U}_{t-2}(\alpha, \delta)] \\ \underline{N}(\alpha') [\underline{S}_t(\alpha', \delta) + \underline{T}_{t-1}(\alpha', \delta) + \underline{U}_{t-2}(\alpha', \delta)] \end{bmatrix}. \quad (2.30b)$$

For a nontrivial solution of $\underline{q}_0^{(k)}$, we see from (2.29a) that

$$\| \underline{K}(\delta) \| = 0. \quad (2.31)$$

Thus δ is the eigenvalue of the matrix \underline{K} and $\underline{q}_0^{(0)}$ is the associated eigenvector. With δ and $\underline{q}_0^{(0)}$ obtained from (2.31) and (2.29a), (2.29b) provides $\underline{q}_0^{(k)}$ for $k \geq 1$ and (2.27a) gives $\underline{q}_t^{(k)}$ for $1 \leq t \leq k$.

When δ is a simple root of (2.31), $\underline{q}_0^{(0)}$ obtained from (2.29) is unique up to an arbitrary multiplicative constant. $\underline{q}_0^{(k)}$ for $k > 1$ obtained from (2.27a) is unique in terms of $\underline{q}_0^{(0)}$ provided $\delta+k$ is not a root of (2.31). Therefore, when δ is a simple root and $\delta+k$ is not a root of (2.31), the eigenfunction ϕ associated with δ is unique up to an arbitrary multiplicative constant. If δ is a multiple root, say a double root of (2.31), and (2.29a) provides two independent $\underline{q}_0^{(0)}$, we would have two independent eigenfunctions each of which is unique up to a multiplicative constant provided $\delta+k$ is not a root of (2.31). When $\delta+k$ is also an eigenvalue of \underline{K} , we see from (2.29b) that a solution for $\underline{q}_0^{(k)}$ exists if and only if

$$\underline{L}^T \sum_{t=1}^k \underline{W}_t(\delta+k) \underline{q}_t^{(k)} = 0 \quad (2.32a)$$

(Hilderbrand 1954) where the superscript T denotes the transpose and \underline{L} is the left eigenvector of $\underline{K}(\delta+k)$

$$\underline{L}^T \underline{K}(\delta+k) = 0. \quad (2.32b)$$

If (2.32a) holds, $g_0^{(k)}$ exists but is not unique. However, the nonunique portion of $g_0^{(k)}$ can be ignored because that portion is represented by the eigenfunction associated with the eigenvalue $\delta+k$. An example of this case in a related problem can be found in Dempsey (1981), Zwiers et al (1982), and Ting and Chou (1985).

If $\delta+k$ is an eigenvalue of \underline{K} and (2.32a) does not hold, a solution for $g_0^{(k)}$ does not exist. In this case, the solution for $\phi^{(k)}$ cannot be given by equation (2.9b). Instead, we use the following modified solution:

$$\phi^{(k)} = \frac{\partial}{\partial \delta} \sum_{t=0}^k A_t^{(k)} x_t z^{\delta+k-t+2} \quad (2.33)$$

in which $A_t^{(k)}$ is now assumed to depend on δ . This case will be discussed in Chapter III. Equation (2.33) can also be used for second independent solution when δ is a double root of (2.31) but (2.29a) provides only one independent $g_0^{(0)}$.

We see from equations (2.22) and (2.28) that for each eigenvalue δ , the stress has the terms $\rho^\delta f_0(\psi, \delta)$, $\rho^{\delta+1} f_1(\psi, \delta)$, $\rho^{\delta+2} f_2(\psi, \delta)$, ... Thus the eigenfunction associated with an eigenvalue has infinite terms for axisymmetric deformations. We also see that if $\text{Re}(\delta) < 0$, the stress is singular at $\rho = 0$. Thus $\text{Re}(\delta)$ provides the order of singularity.

2.4 DEGENERATE CASE $m_1 = m_2 = 1$

When $p_1 = p_2$, p must be purely imaginary. This follows from equation (2.5b) and the fact that p cannot be real. By equation (2.4b) and (2.5c), we have $m_1 = m_2 = 1$. We cannot have $m_1 = m_2 = -1$ because this would make p real. By setting $m = 1$, the second equality of equation (2.4a) yields

$$(c_{13} + 2c_{44})^2 = c_{11}c_{33} . \quad (2.34)$$

Hence when equation (2.34) is satisfied, $p_2 = p_1$, and we have a degenerate case. The five material constants are now reduced to four by equation (2.34). Introducing the new material constants ν , μ , γ , and β , we let

$$c_{11} = (\lambda + 2\mu) \beta^2, \quad (2.35a)$$

$$c_{33} = (\lambda + 2\mu)/\beta^2, \quad (2.35b)$$

$$c_{44} = \mu, \quad (2.35c)$$

$$c_{13} = \lambda, \quad (2.35d)$$

$$c_{11} - c_{33} = 2\gamma\mu, \quad (2.35e)$$

in which

$$\lambda = 2\mu\nu/(1 - 2\nu) . \quad (2.35f)$$

Equations (2.35) satisfy (2.34). With (2.35), equations (2.5) give

$$p = \beta i, \quad m = 1 . \quad (2.36)$$

For isotropic materials we have $\beta = \gamma = 1$ and ν and μ are the Poisson's ratio and shear modulus, respectively.

In a degenerate case $p_1 = p_2$, the terms associated with $B_t^{(k)}$ and $D_t^{(k)}$ are identical, respectively, to the terms associated with $A_t^{(k)}$ and $C_t^{(k)}$. We therefore need a new solution for $B_t^{(k)}$ and $D_t^{(k)}$. This can be accomplished by replacing the coefficients of $B_t^{(k)}$ and $D_t^{(k)}$ by their derivatives with respect to p_1 and \bar{p}_2 (Ting and Chou 1981B, Ting 1982). Thus, for instance, equation (2.14a) becomes

$$\begin{aligned}\phi^{(k)} = & \sum_{t=0}^k \{A_t^{(k)} x^t z^{\delta+k-t+2} + C_t^{(k)} x^t \bar{z}^{\delta+k-t+2}\} \\ & + \sum_{t=0}^k \{B_t^{(k)} z x^t z^{\delta+k-t+1} + D_t^{(k)} z x^t \bar{z}^{\delta+k-t+1}\} (\delta+k-t+2)\end{aligned}\quad (2.37)$$

where, since $p_1 = p_2$, we have omitted the subscripts 1 and 2 for Z and \bar{Z} . Similarly, equations (2.17) and (2.18) are replaced by (noting that \mathbf{m} in the $B_t^{(k)}$ and $D_t^{(k)}$ terms must also be differentiated with respect to p by using equation (2.4b)),

$$\begin{aligned}u_r^{(k)} = & \sum_{t=0}^k [A_t^{(k)}(\delta+k-t+2) + A_{t+1}^{(k)}(t+1)] x^t z^{\delta+k-t+1} \\ & + \sum_{t=0}^k [B_t^{(k)}(\delta+k-t+2) + B_{t+1}^{(k)}(t+1)] (\delta+k-t+1) z x^t z^{\delta+k-t} + \dots\end{aligned}\quad (2.38a)$$

$$\begin{aligned}u_z^{(k)} = & \sum_{t=0}^k [A_t^{(k)} \beta_i - B_t^{(k)} (3-4\nu)] (\delta+k-t+2) x^t z^{\delta+k-t+1} \\ & + \sum_{t=0}^k B_t^{(k)} \beta_i (\delta+k-t+2) (\delta+k-t+1) z x^t z^{\delta+k-t} + \dots\end{aligned}\quad (2.38b)$$

$$\begin{aligned} \frac{1}{2\mu} \sigma_r^{(k)} = & \sum_{t=0}^k \left\{ (A_t^{(k)} \beta^2 - B_t^{(k)} 2\nu \beta i) (\delta+k-t+2) (\delta+k-t+1) \right. \\ & + A_{t+1}^{(k)} 2\gamma(t+1) (\delta+k-t+1) + A_{t+2}^{(k)} \gamma(t+2) (t+1) \left. \right\} x^t z^{\delta+k-t} \\ & + \sum_{t=0}^k \left[B_t^{(k)} \beta^2 (\delta+k-t+2) (\delta+k-t+1) + B_{t+1}^{(k)} 2\gamma(t+1) (\delta+k-t+1) \right. \\ & \left. + B_{t+2}^{(k)} \gamma(t+2) (t+1) \right] (\delta+k-t) zx^t z^{\delta+k-t-1} + \dots \end{aligned} \quad (2.39a)$$

$$\begin{aligned} \frac{1}{2\mu} \sigma_\theta^{(k)} = & \sum_{t=0}^k \left\{ [A_t^{(k)} (\beta^2 - \gamma) - B_t^{(k)} 2\nu \beta i] (\delta+k-t+2) (\delta+k-t+1) \right. \\ & - A_{t+1}^{(k)} 2\gamma(t+1) (\delta+k-t+1) - A_{t+2}^{(k)} \gamma(t+2) (t+1) \left. \right\} x^t z^{\delta+k-t} \\ & + \sum_{t=0}^k \left[B_t^{(k)} (\beta^2 - \gamma) (\delta+k-t+2) (\delta+k-t+1) - B_{t+1}^{(k)} 2\gamma(t+1) (\delta+k-t+1) \right. \\ & \left. - B_{t+2}^{(k)} \gamma(t+2) (t+1) \right] (\delta+k-t) zx^t z^{\delta+k-t-1} + \dots \end{aligned} \quad (2.39b)$$

$$\begin{aligned} \frac{1}{2\mu} \sigma_z^{(k)} = & \sum_{t=0}^k - [A_t^{(k)} + B_t^{(k)} 2(1-\nu) i \beta^{-1}] (\delta+k-t+2) (\delta+k-t+1) x^t z^{\delta+k-t} \\ & - \sum_{t=0}^k B_t^{(k)} (\delta+k-t+2) (\delta+k-t+1) (\delta+k-t) zx^t z^{\delta+k-t-1} + \dots \end{aligned} \quad (2.39c)$$

$$\begin{aligned} \frac{1}{2\mu} \sigma_{rz}^{(k)} = & \sum_{t=0}^k \left\{ [A_t^{(k)} \beta i - B_t^{(k)} (1-2\nu)] (\delta+k-t+2) \right. \\ & + [A_{t+1}^{(k)} \beta i - B_{t+1}^{(k)} (1-2\nu)] (t+1) \left. \right\} (\delta+k-t+1) x^t z^{\delta+k-t} \\ & + \sum_{t=0}^k [B_t^{(k)} (\delta+k-t+2) + B_{t+1}^{(k)} (t+1)] (\delta+k-t+1) (\delta+k-t) \beta i zx^t z^{\delta+k-t-1} + \dots \end{aligned} \quad (2.39d)$$

In equations (2.38) and (2.39), the dots stand for the $C_t^{(k)}$ and $D_t^{(k)}$ terms that are obtained from the $A_t^{(k)}$ and $B_t^{(k)}$ terms by replacing βi and z by $-\beta i$ and \bar{z} , respectively. Equations (2.20)–(2.32) remain valid except (2.25), which are replaced by

$$\begin{aligned} \tilde{S}_t(\psi, \delta) & \quad (2.40a) \\ = 2\mu(\delta-t+2)(\delta-t+1) & \begin{bmatrix} \beta^2 & -2\nu\beta i + \beta^2(\delta-t)\zeta^{-1}\sin\psi & * & * \\ \beta i & -(1-2\nu) + (\delta-t)\beta i \zeta^{-1}\sin\psi & * & * \\ -1 & -2(1-\nu)i\beta^{-1} - (\delta-t)\zeta^{-1}\sin\psi & * & * \end{bmatrix} \tilde{Q}_t(\psi, \delta), \end{aligned}$$

$$\underline{I}_t(\psi, \delta) \quad (2.40b)$$

$$= 2\mu(t+1)(\delta-t+1) \begin{bmatrix} 2\gamma & 2\gamma(\delta-t)\xi^{-1}\sin\psi & * & * \\ \beta_i & -(1-2\nu)+(\delta-t)\beta_i\xi^{-1}\sin\psi & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{Q}_t(\psi, \delta),$$

$$\underline{U}_t(\psi, \delta) \quad (2.40c)$$

$$= 2\mu\gamma(t+2)(t+1) \begin{bmatrix} 1 & (\delta-t)\xi^{-1}\sin\psi & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{Q}_t(\psi, \delta).$$

$\underline{Q}_t(\psi, \delta)$ is obtained from (2.26) with $\xi_1 = \xi_2 = \xi$. The third and fourth columns of the matrices in (2.40) are obtained from the first and second columns by replacing β_i and ξ by $-\beta_i$ and ξ , respectively. Equation (2.27) remains valid because the order of the differentiation with respect to p and x or z can be interchanged.

2.5 PARTICULAR SOLUTION FOR THE DISPLACEMENT OF THE SINGULAR POINT

For a two-dimensional problem, the displacement of a singular point can be ignored for the singularity analysis. For an axisymmetric deformation, one cannot ignore the displacement u_r of a singular point. We therefore consider $(u_r, u_z) = (u_0, 0)$ at the singular point $(r, z) = (a, 0)$ where u_0 is a constant. A particular solution that yields this displacement is

$$u_r = u_0 r/a, \quad u_z = 0, \quad (2.40a)$$

$$\sigma_r = \sigma_\theta = u_0(c_{1,1} + c_{1,2})/a, \quad \sigma_z = 2u_0 c_{1,3}/a, \quad \sigma_{rz} = 0. \quad (2.40b)$$

Let

$$\underline{\underline{\sigma}}_0 = \frac{u_0}{a} \begin{bmatrix} c_{11} + c_{12} \\ 0 \\ 2c_{13} \end{bmatrix}. \quad (2.42)$$

To satisfy the stress-free boundary conditions, equation (2.20b), we superimpose equation (2.42) to (2.23) with $\delta = 0$ and write the stress as

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}_0 + \sum_{k=0}^{\infty} \rho^k \sum_{t=0}^k \{ \underline{\underline{S}}_t(\psi, k) \underline{\underline{q}}_t^{(k)} + \underline{\underline{T}}_t(\psi, k) \underline{\underline{q}}_{t+1}^{(k)} + \underline{\underline{U}}_t(\psi, k) \underline{\underline{q}}_{t+2}^{(k)} \}. \quad (2.43)$$

Equation (2.20b) now provides the following equations for $\underline{\underline{q}}_0^{(k)}$

$$\underline{\underline{K}}(k) \underline{\underline{q}}_0^{(k)} = \underline{\underline{b}}^{(k)}, \quad (k = 0, 1, 2, \dots), \quad (2.44)$$

in which

$$\underline{\underline{b}}^{(0)} = - \begin{bmatrix} \underline{\underline{N}}(\alpha) \\ \underline{\underline{N}}(\alpha') \end{bmatrix} \underline{\underline{\sigma}}_0, \quad (2.45a)$$

$$\underline{\underline{b}}^{(k)} = - \sum_{t=1}^k \underline{\underline{W}}_t(k) \underline{\underline{q}}_t^{(k)}, \quad (k \geq 1), \quad (2.45b)$$

and $\underline{\underline{W}}_t(k)$ is defined in equation (2.30b). Equation (2.44) has a unique solution for $\underline{\underline{q}}_0^{(k)}$ if k is not an eigenvalue of $\underline{\underline{K}}$. If k is, then the discussion presented near the end of Section 2.3 applies here.

Chapter III

MODIFIED EIGENFUNCTIONS

3.1 MODIFIED SOLUTION

When δ is a root of (2.29a), let η be the smallest positive integer for which $\delta+\eta$ is also a root of (2.29a). Equation (2.29b) for $k = \eta$ is

$$\underline{K}(\delta+\eta) \underline{g}_0^{(\eta)} = - \sum_{t=1}^{\eta} \underline{w}_t(\delta+\eta) \underline{g}_t^{(\eta)}. \quad (3.1)$$

This has a solution for $\underline{g}_0^{(\eta)}$ unless

$$\underline{L}^T \sum_{t=1}^{\eta} \underline{w}_t(\delta+\eta) \underline{g}_t^{(\eta)} \neq 0, \quad (3.2)$$

where \underline{L} is the left eigenvector

$$\underline{L}^T \underline{K}(\delta+\eta) = \underline{0}. \quad (3.3)$$

If (3.2) holds, a solution for $\underline{g}_0^{(\eta)}$ does not exist and the expansion for ϕ given by (2.9) is not valid.

To obtain a valid expansion when (3.2) holds, we notice that if ϕ given by (2.9) satisfies (2.8) so does $\partial\phi/\partial\delta$. Therefore, in place of (2.14a) we use

$$\phi(k) = \frac{\partial}{\partial\delta} \left\{ \sum_{t=0}^k A_t^{(k)} x^t z^{\delta+k-t+2} + \dots \right\} \quad (3.4)$$

in which $A_t^{(k)} \dots D_t^{(k)}$ are now regarded as functions of δ (Zwiers et al 1982). If we carry out the differentiation in (3.4), we will have terms of the form $x^a z^b$ as well as $x^a z^b (\ln z)$. By substituting (3.4) into (2.9a) and then into (2.8), the coefficients of $x^a z^b$ and $x^a z^b (\ln z)$ must vanish. The latter leads to (2.27). The former leads to the following equations which can also be obtained by differentiating (2.27) with respect to δ :

$$\begin{aligned} \underline{q}_t^{(k)} &= \frac{2t-1}{2t} \underline{q}_{t-1}^{(k-1)} + \frac{1}{2(\delta+k-t+2)} [t \underline{q}_t^{(k-1)} - (t+1) \underline{q}_{t+1}^{(k)}] \\ &\quad - \frac{1}{2(\delta+k-t+2)^2} [t \underline{q}_t^{(k-1)} - (t+1) \underline{q}_{t+1}^{(k)}] , \\ &\quad \text{for } t = k, k-1, \dots, 1 , \end{aligned} \quad (3.5a)$$

$$\underline{q}_t^{(k)} = 0 , \quad \text{for } t > k , \quad (3.5b)$$

where the prime denotes differentiation with respect to δ . Hence the only unknowns are $\underline{q}_0^{(k)}$ and $\underline{q}_0^{(k)}$, ($k = 0, 1, 2 \dots$).

When (3.2) holds and η is the smallest positive integer for which $\delta+\eta$ is also a root of (2.29a), we may choose

$$\underline{q}_t^{(k)} = 0 , \quad \text{for } k < \eta . \quad (3.6)$$

Substituting (3.4) into (2.1) and (2.2) and carrying out the differentiation, we obtain new expressions for the displacements and stresses. This is presented in Appendix B.

for the degenerate case in which $p_1 = p_2$, we use instead of (3.4)

$$\begin{aligned} \phi^{(k)} &= \frac{\partial}{\partial \delta} \left\{ \sum_{t=0}^k [A_t^{(k)} x^t z^{\delta+k-t+2} + C_t^{(k)} x^t z^{-\delta+k-t+2}] \right. \\ &\quad \left. + \sum_{t=0}^k [B_t^{(k)} z x^t z^{\delta+k-t+1} + D_t^{(k)} z x^t z^{-\delta+k-t+1}] (\delta+k-t+2) \right\} \end{aligned} \quad (3.7)$$

Following the same argument, we obtain new expressions for the displacements and stresses. This is also presented in Appendix B.

3.2 DETERMINATION OF $\underline{\underline{q}}_0^{(k)}$ AND $\underline{\underline{q}}_0'^{(k)}$

The satisfaction of the stress-free boundary conditions leads to the following system of equations which can also be obtained by applying the operator $\rho^{\delta+k}(\ln \rho + \partial/\partial \delta)$ to (2.29) and setting the coefficients of $\rho^{\delta+k}$ and $\rho^{\delta+k} \ln \rho$ to zero. Thus we obtain from (2.29a)

$$\underline{\underline{K}}(\delta) \underline{\underline{q}}_0^{(0)} = \underline{\underline{0}}, \quad (3.8a)$$

$$\underline{\underline{K}}'(\delta) \underline{\underline{q}}_0^{(0)} + \underline{\underline{K}}(\delta) \underline{\underline{q}}_0'^{(0)} = \underline{\underline{0}}, \quad (3.8b)$$

and from (2.29b)

$$\underline{\underline{K}}(\delta+k) \underline{\underline{q}}_0^{(k)} = - \sum_{t=1}^k \underline{\underline{w}}_t(\delta+k) \underline{\underline{q}}_t^{(k)}, \quad (k \geq 1), \quad (3.8c)$$

$$\underline{\underline{K}}'(\delta+k) \underline{\underline{q}}_0^{(k)} + \underline{\underline{K}}(\delta+k) \underline{\underline{q}}_0'^{(k)} = - \sum_{t=1}^k \{ \underline{\underline{w}}_t'(\delta+k) \underline{\underline{q}}_t^{(k)} + \underline{\underline{w}}_t(\delta+k) \underline{\underline{q}}_t'^{(k)} \}, \quad (3.8d)$$

where $\underline{\underline{K}}$ and $\underline{\underline{w}}_t$ are defined in (2.30).

Combining (3.5) and (3.8) with (3.6), we notice that (3.5) and (3.8) are the same as (2.27) and (2.29) for $k < \eta$, respectively, except $\underline{\underline{q}}_0'^{(k)}$ now assumes the role of $\underline{\underline{q}}_0^{(k)}$.

For $k = \eta$ the problem reduces to solving the following system of equations

$$\underline{\underline{K}}(\delta+\eta) \underline{\underline{q}}_0^{(\eta)} = \underline{\underline{0}}, \quad (3.9a)$$

$$\underline{K}'(\delta+\eta) \underline{q}_0^{(\eta)} + \underline{K}(\delta+\eta) \underline{q}_0'(\eta) = - \sum_{t=1}^{\eta} \underline{w}_t(\delta+\eta) \underline{q}_t'(\eta). \quad (3.9b)$$

Equations (3.9) have an unique solution for $\underline{q}_0^{(\eta)}$ if (Dempsey and Sinclair 1979)

$$d^i \|\underline{K}(\delta)\| / d\delta^i \neq 0, \quad (i = n - m), \quad (3.10)$$

where n and m are, respectively, the order and rank of \underline{K} .

It is rather difficult to prove or disprove equation (3.10) analytically or numerically. Instead, we will regard (3.9) as a system of 8 equations for $\underline{q}_0^{(\eta)}$ and $\underline{q}_0'(\eta)$, and solve the system numerically.

For $k > \eta$, (3.8c) and (3.8d) give $\underline{q}_0^{(k)}$ and $\underline{q}_0'^{(k)}$.

Chapter IV
APPLICATIONS TO COMPOSITES

4.1 SINGULAR POINT IN COMPOSITE MATERIALS

We now consider the axisymmetric composite whose cross section is shown in Fig.2. The two materials with axisymmetric interface SQ, RP are assumed to be transversely isotropic with the z-axis being the axis of symmetry. The interface makes an angle ψ_1 with the $z = 0$ plane. The region SMRN is void and ψ_1 and ψ_2 are the angles the two free surfaces RM and RN make with the $z = 0$ plane.

Since equations (2.9) is applicable to each material, we will use the subscript 1 or 2 separated by a comma to identify the quantity which is associated with material 1 or 2. From equation (2.27) we notice that the undetermined constants for material 1 are $A_{0,1}^{(k)} \dots D_{0,1}^{(k)}$ while that for material 2 are $A_{0,2}^{(k)} \dots D_{0,2}^{(k)}$. The eigenvalue δ is the same for both materials.

4.2 DETERMINATION OF δ AND $A_{0,S}^{(k)} \dots D_{0,S}^{(k)}$

Using (2.20b), the traction-free boundary conditions at angles $\psi = \psi_1$ and ψ_2 are

$$\mathbf{N}(\psi_s) \mathbf{\sigma}_{s,S} = \mathbf{0} \quad , \quad (s=1,2) \quad , \quad (4.1)$$

where

$$\mathbf{N}(\psi) = \begin{bmatrix} \sin\psi & -\cos\psi & 0 \\ 0 & \sin\psi & -\cos\psi \end{bmatrix} \quad , \quad (4.2)$$

$$\underline{\sigma}_{s,s} = \sum_{k=0}^{\infty} \underline{\sigma}_{s,s}^{(k)}, \quad \underline{\sigma}_{s,s}^{(k)} = \begin{bmatrix} \sigma_{r,s}^{(k)} \\ \sigma_{rz,s}^{(k)} \\ \sigma_{z,s}^{(k)} \end{bmatrix}. \quad (4.3)$$

The interface continuity conditions at angle $\psi = \psi_3$ are

$$N(\psi_3) \underline{\sigma}_{s,1} - N(\psi_3) \underline{\sigma}_{s,2} = \underline{0}, \quad (4.4)$$

$$\underline{u}_{s,1} - \underline{u}_{s,2} = \underline{0}, \quad (4.5)$$

where

$$\underline{u}_{s,s} = \sum_{k=0}^{\infty} \underline{u}_{s,s}^{(k)}, \quad \underline{u}_{s,s}^{(k)} = \begin{bmatrix} u_{r,s}^{(k)} \\ u_{rz,s}^{(k)} \\ u_{z,s}^{(k)} \end{bmatrix}, \quad (s=1,2). \quad (4.6)$$

Substitution of equations (2.17) and (2.18) into equations (4.1), (4.4) and (4.5) yields the following system of recurrent equations

$$K(\delta) \underline{q}_0^{(0)} = \underline{0}, \quad (4.7a)$$

$$K(\delta+k) \underline{q}_0^{(k)} = - \sum_{t=1}^k w_t(\delta+k) \underline{q}_t^{(k)}, \quad (k \geq 1), \quad (4.7b)$$

where

$$\underline{q}_t^{(k)} = \begin{bmatrix} q_{t,1}^{(k)} \\ q_{t,2}^{(k)} \end{bmatrix}, \quad (t = 0, 1, 2, \dots), \quad (4.8)$$

in which the elements of $\underline{q}_{t,s}^{(k)}$ are $A_{t,s}^{(k)}, \dots, D_{t,s}^{(k)}$ ($s = 1, 2$), and

$$\underline{K}(\delta) = \begin{bmatrix} \underline{N}(\psi_1) \underline{S}_{0,1}(\psi_1, \delta) & 0 \\ \underline{N}(\psi_3) \underline{S}_{0,1}(\psi_3, \delta) & -\underline{N}(\psi_3) \underline{S}_{0,2}(\psi_3, \delta) \\ \underline{G}_{0,1}(\psi_3, \delta) & -\underline{G}_{0,2}(\psi_3, \delta) \\ 0 & \underline{N}(\psi_2) \underline{S}_{0,2}(\psi_2, \delta) \end{bmatrix} \quad (4.9)$$

$$\underline{w}_t(\delta) = \begin{bmatrix} \underline{N}(\psi_1) [\underline{S}_{t,1}(\psi_1, \delta) + \underline{I}_{t-1,1}(\psi_1, \delta) + \underline{U}_{t-2,1}(\psi_1, \delta)] \\ \underline{N}(\psi_3) [\underline{S}_{t,1}(\psi_3, \delta) + \underline{I}_{t-1,1}(\psi_3, \delta) + \underline{U}_{t-2,1}(\psi_3, \delta)] \\ \underline{G}_{t,1}(\psi_3, \delta) + \underline{v}_{t-1,1}(\psi_3, \delta) \\ 0 \\ 0 \\ -\underline{N}(\psi_3) [\underline{S}_{t,1}(\psi_3, \delta) + \underline{I}_{t-1,1}(\psi_3, \delta) + \underline{U}_{t-2,1}(\psi_3, \delta)] \\ -[\underline{G}_{t,2}(\psi_3, \delta) + \underline{v}_{t-1,2}(\psi_3, \delta)] \\ \underline{N}(\psi_2) [\underline{S}_{t,2}(\psi_2, \delta) + \underline{I}_{t-1,2}(\psi_2, \delta) + \underline{U}_{t-2,2}(\psi_2, \delta)] \end{bmatrix} \quad (4.10)$$

In (4.9) and (4.10), $\underline{S}_{t,s}$, $\underline{I}_{t,s}$, $\underline{U}_{t,s}$ are defined in equations (2.25) and \underline{G}_t and \underline{v}_t are given by

$$\underline{G}_t(\psi, \delta) = (\delta-t+2) \begin{bmatrix} 1 & 1 & 1 & 1 \\ \underline{m}_1 p_1 & \underline{m}_2 p_2 & \bar{\underline{m}}_1 \bar{p}_1 & \bar{\underline{m}}_2 \bar{p}_2 \end{bmatrix} \underline{Q}_t(\psi, \delta+1), \quad (4.11a)$$

$$\underline{v}_t(\psi, \delta) = (t+1) \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{Q}_t(\psi, \delta+1), \quad (4.11b)$$

where $\underline{Q}_t(\psi, \delta)$ is the same as the one in (2.26).

It should be pointed out that the recurrent relation for $\underline{g}_t^{(k)}$ given in (2.27) applies to $\underline{g}_t^{(k)}$ in (4.8). Therefore the problem reduces to the determination of δ and $A_{0,s}^{(k)} \dots D_{0,s}^{(k)}$ ($s = 1, 2$). For nontrivial solution of $\underline{g}_0^{(0)}$ we must have

$$\| \underline{K}(\delta) \| = 0. \quad (4.12)$$

This provides the eigenvalue δ . For each eigenvalue δ , (4.7a) gives $\underline{q}_0^{(0)}$ while (2.27) and (4.7b) furnish $\underline{q}_0^{(k)}$ ($k = 1, 2, \dots$). The discussion on uniqueness of solutions for $\underline{q}_0^{(k)}$ presented in section 2.3 applies here.

4.3 DEGENERATE CASE

When $p_1 = p_2$ holds for one of the two materials or for both materials, we use the expressions for displacements and stresses in (2.38) and (2.39). Once again, the stress-free boundary conditions and the interface continuity conditions yield the system of recurrent equations (4.7). Equations (4.9)-(4.10) hold in which $\underline{S}_{t,s}$, $\underline{I}_{t,s}$, $\underline{U}_{t,s}$ are defined in equations (2.40) if material s ($s = 1, 2$) is degenerate, while (4.11) is replaced by

$$\underline{G}_t(\psi, \delta) = (\delta-t+2) \begin{bmatrix} 1 & (\delta-t+1)\zeta^{-1}\sin\psi & * & * \\ \beta_i & \beta_i(\delta-t+1)\zeta^{-1}\sin\psi-3+4\nu & * & * \end{bmatrix} \underline{Q}_t(\psi, \delta+1), \quad (4.14a)$$

$$\underline{v}_t(\psi, \delta) = (t+1) \begin{bmatrix} 1 & (\delta-t+1)\zeta^{-1}\sin\psi & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{Q}_t(\psi, \delta+1). \quad (4.14b)$$

$\underline{Q}_t(\psi, \delta)$ is obtained from equation (2.26) with $\zeta_1 = \zeta_2 = \zeta$. As in equation (2.40) the third and fourth columns of the matrices in equations (4.14) are obtained from the first and second columns by replacing β_i and ζ by $-\bar{\beta}_i$ and $\bar{\zeta}$, respectively.

4.4 NUMERICAL EXAMPLES

We present two examples in this section. In both examples $\psi_1 = -180^\circ$, $\psi_2 = 90^\circ$ and $\psi_3 = -90^\circ$ are taken.

In the first example the material 1 and 2 are both isotropic so that $\beta = \gamma = 1$. We use $\nu = 0.38$, $\mu = 0.3 \times 10^6$ psi for material 1 and $\nu = 0.45$, $\mu = 0.3448 \times 10^6$ psi for material 2. Two negative δ 's are obtained:

$$\delta_1 = -0.432087, \quad \delta_2 = -0.073520.$$

δ_1 is the same as p_1 of plane strain problem obtained by Bogy (1971). The coefficients of order zero which are complex-valued are as follows.

For δ_1 ,

$$\begin{aligned} A_{0,1}^{(0)} &= (0.3446 - 0.6786i)c_1, \quad B_{0,1}^{(0)} = (0.4098 + 0.9122i)c_1, \\ A_{0,2}^{(0)} &= (0.1444 - 0.4991i)c_1, \quad B_{0,2}^{(0)} = (0.4313 + 0.8995i)c_1, \end{aligned}$$

and for δ_2 ,

$$\begin{aligned} A_{0,1}^{(0)} &= (0.1744 - 0.2567i)c_2, \quad B_{0,1}^{(0)} = (0.8936 - 0.0208i)c_2, \\ A_{0,2}^{(0)} &= (0.1576 - 0.1539i)c_2, \quad B_{0,2}^{(0)} = (0.9986 - 0.0533i)c_2, \end{aligned}$$

where c_1 and c_2 are arbitrary multiplicative constants. C's and D's are the complex conjugate of A's and B's respectively, because δ_1 and δ_2 are real.

In the second example material 1 is replaced by a transversely isotropic material whose material constants are (with unit 10^6 psi)

$$\begin{aligned} c_{11} &= 2.152, \quad c_{12} = 0.5524, \quad c_{13} = 0.8115, \\ c_{33} &= 34.49, \quad c_{44} = 0.8. \end{aligned}$$

The corresponding p's and m's are

$$p_1 = 0.1551i, \quad p_2 = 1.611i, \\ m_1 = 55.02, \quad m_2 = 0.01816.$$

Again, two negative δ 's are obtained:

$$\delta_1 = -0.484629, \quad \delta_2 = -0.299609.$$

The coefficients of order zero are as follows. For δ_1 ,

$$A_{0,1}^{(0)} = (0.0486-0.0015i)c_1, \quad B_{0,1}^{(0)} = (-0.2669+0.1983i)c_1, \\ A_{0,2}^{(0)} = (-0.0429+0.4953i)c_1, \quad B_{0,2}^{(0)} = (0.0351-0.9994i)c_1,$$

and for δ_2 ,

$$A_{0,1}^{(0)} = (0.0405+0.0399i)c_2, \quad B_{0,1}^{(0)} = (0.0330-0.5511i)c_2, \\ A_{0,2}^{(0)} = (0.2865-0.2931i)c_2, \quad B_{0,2}^{(0)} = (0.9650+0.2621i)c_2.$$

Since $\delta+k$ where k is an arbitrary integer is not a root of (4.12) for both cases, the solutions are unique up to the arbitrary constants c_1 and c_2 .

The stress distribution obtained from the first term of the eigenfunctions associated with δ_1 and δ_2 are plotted in Fig.3 - Fig.6. We normalize the stress by deviding by the singular factor ρ^δ and a multiplicative constant c to make the maximum stress equal to 1.

Chapter V

EIGENFUNCTIONS AT AN INTERFACE CRACK WITH A CONTACT ZONE

5.1 UNREALISTIC PHENOMENON

We have discussed in Chapter IV the stress singularities at a singular point of an axisymmetric composite in which the free surfaces and the interface surface intersect. When the two free surfaces make the same angle with the plane $z = 0$, the free surfaces form an interface crack, Fig. 7. When the singularity δ is a complex number, an oscillatory phenomenon in displacement near the crack tip occurs and the two free surfaces inter-penetrate each other. To avoid the unrealistic phenomenon, we assume that a contact zone is presented near the crack tip. In a real composite, the crack surfaces near the interface crack tip may, under an external load, open or close with or without friction. The associated problem for isotropic composites was studied for frictionless contact and for contact with friction by Comninou (1977A and 1977B). Wang (1983) studied the partially closed interface crack for anisotropic materials but the contact region is assumed to be frictionless. We will use the asymptotic solutions (2.17), (2.18), (2.38) and (2.39) to study the stress singularities at the both ends of the contact zone.

5.2 SINGULARITIES AT ENDS OF CONTACT ZONE IN INTERFACE CRACK

In Fig. 7, AB is the contact zone, AC and AD are free surfaces, and BE is the interface. We will call the singularity analyses around point

A and point B, respectively, Case A and Case B. It should be noted that there is only one independent angle in case A whereas there are two in case B. Let θ be the angle of orientation of the crack. In case A, $\psi_3 = \theta$, $\psi_1 = \theta + \pi$ and $\psi_2 = \theta - \pi$. In case B, ψ_3 is arbitrary.

Using the same notations we have used before, we have the following boundary conditions for Case A:

$$\underline{N}(\psi_s) \underline{\sigma}_{s,s} = 0, \quad (s=1,2), \quad (5.1a)$$

$$\underline{N}(\psi_3) \underline{\sigma}_{3,1} - \underline{N}(\psi_3) \underline{\sigma}_{3,2} = 0, \quad (5.1b)$$

$$\underline{J}(\psi_3) \underline{u}_{s,1} - \underline{J}(\psi_3) \underline{u}_{s,2} + \underline{H}(\psi_3) \underline{\sigma}_{s,1} = 0, \quad (5.1c)$$

where

$$\underline{J}(\psi) = \begin{bmatrix} -\sin\psi & \cos\psi \\ 0 & 0 \end{bmatrix}, \quad (5.2a)$$

$$\underline{H}(\psi)^T = \begin{bmatrix} 0 & \sin\psi[\cos\psi + \tau(\operatorname{sgn} t_s)\sin\psi] \\ 0 & -\cos2\psi - \tau(\operatorname{sgn} t_s)\sin2\psi \\ 0 & -\cos\psi[\sin\psi - \tau(\operatorname{sgn} t_s)\cos\psi] \end{bmatrix} \quad (5.2b)$$

In (5.2b) τ is the coefficient of friction and $\operatorname{sgn} t_s$ stands for the sign of shear traction t_s .

Substitution of (2.17) and (2.18) or (2.38) and (2.39) into (5.1) yields a system of recurrent equations similar to (4.7) in which \underline{K} and \underline{W}_t have the expressions:

$$\underline{K}(\delta) = \begin{bmatrix} \underline{N}(\psi_1) \underline{S}_{0,1}(\psi_1, \delta) & 0 \\ \underline{N}(\psi_3) \underline{S}_{0,1}(\psi_3, \delta) & -\underline{N}(\psi_3) \underline{S}_{0,2}(\psi_3, \delta) \\ \underline{J}(\psi_3) \underline{G}_{0,1}(\psi_3, \delta) + \underline{H}(\psi_3) \underline{S}_{0,1}(\psi_3, \delta) & -\underline{J}(\psi_3) \underline{G}_{0,2}(\psi_3, \delta) \\ 0 & \underline{N}(\psi_2) \underline{S}_{0,2}(\psi_2, \delta) \end{bmatrix}, \quad (5.3a)$$

$$\tilde{W}_t(\delta) = \begin{bmatrix} \tilde{N}(\psi_1) \tilde{E}_{t,1}(\psi_1, \delta) & 0 \\ \tilde{N}(\psi_3) \tilde{E}_{t,1}(\psi_3, \delta) & -\tilde{N}(\psi_3) \tilde{E}_{t,2}(\psi_3, \delta) \\ \tilde{J}(\psi_3) \tilde{F}_{t,1}(\psi_3, \delta) + \tilde{H}(\psi_3) \tilde{E}_{t,1}(\psi_3, \delta) & -\tilde{J}(\psi_3) \tilde{F}_{t,2}(\psi_3, \delta) \\ 0 & \tilde{N}(\psi_2) \tilde{E}_{t,2}(\psi_2, \delta) \end{bmatrix}. \quad (5.3b)$$

In (5.3b),

$$\tilde{E}_{t,i}(\psi, \delta) = \tilde{S}_{t,i}(\psi, \delta) + \tilde{T}_{t-1,i}(\psi, \delta) + \tilde{U}_{t-2,i}(\psi, \delta), \quad (5.4a)$$

$$\tilde{F}_{t,i}(\psi, \delta) = \tilde{G}_{t,i}(\psi, \delta) + \tilde{V}_{t-1,i}(\psi, \delta), \quad (5.4b)$$

in which $\tilde{S}_{t,s}$, $\tilde{T}_{t,s}$, $\tilde{U}_{t,s}$, $\tilde{G}_{t,s}$ and $\tilde{V}_{t,s}$ are defined by (2.25) and (4.11) if the material s is transversely isotropic and by (2.40) and (4.14) if material s is a degenerated material.

For Case B we have the following boundary conditions:

$$\tilde{N}(\psi_1) \tilde{\sigma}_{z,1} - \tilde{N}(\psi_2) \tilde{\sigma}_{z,2} = 0, \quad (5.5a)$$

$$\tilde{J}(\psi_1) \tilde{u}_{z,1} - \tilde{J}(\psi_2) \tilde{u}_{z,2} + \tilde{H}(\psi_1) \tilde{\sigma}_{z,1} = 0, \quad (5.5b)$$

$$\tilde{N}(\psi_3) \tilde{\sigma}_{z,1} - \tilde{N}(\psi_3) \tilde{\sigma}_{z,2} = 0, \quad (5.5c)$$

$$\tilde{u}_{z,1} - \tilde{u}_{z,2} = 0, \quad (5.5d)$$

in which \tilde{J} and \tilde{H} are defined in (5.2). Again, (5.5) yield (4.7) in which

$$\tilde{K}(\delta) = \begin{bmatrix} \tilde{N}(\psi_1) \tilde{S}_{0,1}(\psi_1, \delta) & -\tilde{N}(\psi_2) \tilde{S}_{0,2}(\psi_2, \delta) \\ \tilde{N}(\psi_3) \tilde{S}_{0,1}(\psi_3, \delta) & -\tilde{N}(\psi_3) \tilde{S}_{0,2}(\psi_3, \delta) \\ \tilde{G}_{0,1}(\psi_3, \delta) & -\tilde{G}_{0,2}(\psi_3, \delta) \\ \tilde{J}(\psi_1) \tilde{G}_{0,1}(\psi_1, \delta) + \tilde{H}(\psi_1) \tilde{S}_{0,1}(\psi_1, \delta) & -\tilde{J}(\psi_2) \tilde{G}_{0,2}(\psi_2, \delta) \end{bmatrix}, \quad (5.6a)$$

$$\underline{w}_t(\delta) = \begin{bmatrix} \underline{N}(\psi_1) \underline{E}_{t,1}(\psi_1, \delta) & -\underline{N}(\psi_2) \underline{E}_{t,2}(\psi_2, \delta) \\ \underline{N}(\psi_3) \underline{E}_{t,1}(\psi_3, \delta) & -\underline{N}(\psi_3) \underline{E}_{t,2}(\psi_3, \delta) \\ \underline{F}_{t,1}(\psi_3, \delta) & -\underline{F}_{t,2}(\psi_3, \delta) \\ \underline{J}(\psi_1) \underline{F}_{t,1}(\psi_1, \delta) + \underline{H}(\psi_1) \underline{E}_{t,1}(\psi_1, \delta) & -\underline{J}(\psi_2) \underline{F}_{t,2}(\psi_2, \delta) \end{bmatrix}. \quad (5.6b)$$

It will be shown analytically in Chapter VI that δ is real at both ends of the contact zone. In other words, there is no oscillation of the crack surface displacements near the singular points.

The stresses at both ends of a contact zone without friction are computed numerically for a composite material which is similar to that of Example 2 of Section 4.4. In this case $\delta = -1/2$ (see Section 6.3). We normalize the stresses by dividing by the singular factor ρ^δ and a multiplicative constant c to make the maximum stress equal to 1. The normalized stress distribution obtained from the first term of the eigenfunction for Case A and B are plotted in Fig.8 and Fig.9, respectively.

Chapter VI

SINGULARITIES AT AN INTERFACE CRACK WITH A CONTACT ZONE

6.1 STROH'S FORMALISM

An alternative formulation for the order of singularities δ at an interface crack with a contact zone will be derived in this chapter. The derivation is based on the Stroh formalism (Stroh 1958 and 1962). The Stroh formalism, which has its origin by Eshelby (1953), provides an elegant and powerful method of treating a certain class of two-dimensional anisotropic elasticity problems. Unlike the two-dimensional anisotropic solutions developed by Green and Zerna (1954) which are restricted to plane strain deformations, the Stroh formalism applies to a wide variety of two-dimensional problems in which all three displacement components are non-zero. Also, unlike the widely used Lekhnitskii's approach (Lekhnitskii 1981) which breaks down for orthotropic materials (Ting and Chou 1981A) and requires a special treatment (Ting and Chou 1981B), the Stroh formalism has no limitations except possibly for the degenerate materials in which the eigenvalues of the elasticity constants have a repeated root such as in isotropic materials. The problem with degenerate materials, for which other formalism also have, can be treated separately (Ting 1982). However, the Stroh formalism has since been perfected by Barnett and Lothe (1973 and 1975).

It can be seen from Equations of (2.8) and (2.9) that the first order solution of axisymmetric deformation is the same as that of plane-

strain problem. Therefore, the order of stress singularities δ of axisymmetric deformation at an interface crack with contact can be obtained by Stroh formalism.

6.2 BASIC EQUATIONS

In a fixed rectangular coordinate system (x_1, x_2, x_3) , let the stress-strain law of an anisotropic elastic material be given by

$$\sigma_{ij} = c_{ijkn} u_{k,n} \quad (6.1)$$

where repeated indices imply summation, σ_{ij} , u_k , c_{ijkn} are, respectively, the stress, displacement and elastic constants and a comma stands for partial differentiation. The equation of equilibrium are

$$\sigma_{ij,j} = 0. \quad (6.2)$$

For the purpose of the present analysis, we assume that

$$u_k = a_k z^{\delta+1}/(\delta+1), \quad (6.3a)$$

$$z = x_1 + p x_2, \quad (6.3b)$$

in which p , δ and a_k are constants to be determined. Substituting (6.3) into (6.1) and (6.2) yields

$$\sigma_{ij} = (c_{ijkl} + p c_{ijk2}) a_k z^\delta, \quad (6.4)$$

$$\{Q + p(\tilde{R} + \tilde{R}^T) + p^2 \tilde{T}\} a = 0, \quad (6.5)$$

where the matrices Q , \tilde{R} and \tilde{T} are given by

$$Q_{ik} = c_{ilk1}, \quad R_{ik} = c_{ilk2}, \quad T_{ik} = c_{i2k2}. \quad (6.6)$$

The superscript T stands for the transpose. Equation (6.5) provides three pairs of complex conjugates for the eigenvalue p and the associated eigenvector \underline{a} . If p_m , \underline{a}_m , ($m = 1, 2, \dots, 6$) are the eigenvalues and eigenvectors, we will let

$$p_{m+3} = \bar{p}_m, \quad \underline{a}_{m+3} = \bar{\underline{a}}_m, \quad (6.7)$$

where an overbar denote the complex conjugate. The general solution for \underline{u} as given by (6.3) can be written as

$$\underline{u} = \sum_{m=1}^3 (q_m \underline{a}_m z_m^{\delta+1} + h_m \bar{\underline{a}}_m \bar{z}_m^{\delta+1}) / (\delta+1) \quad (6.8a)$$

in which q_m and h_m are arbitrary constants and

$$z_m = x_1 + p_m x_2 = r(\cos\theta + p_m \sin\theta). \quad (6.8b)$$

In (6.8b), r and θ are the polar coordinates.

Let t_i be the surface traction on a radial plane which makes an angle θ with the x_1 axis. We then have

$$t_i = -\sigma_{i1} \sin\theta + \sigma_{i2} \cos\theta, \quad (6.9)$$

or, using (6.4), (6.5), (6.6) and (6.8b),

$$t_i = \frac{1}{r} \underline{b} z^{\delta+1}, \quad (6.10a)$$

where

$$\underline{b} = (\underline{R} + pT) \underline{a}. \quad (6.10b)$$

The general solution for the surface traction can be written as (Ting 1986)

$$\tilde{t} = \frac{1}{r} \sum_{m=1}^3 (q_m b_m Z_m^{\delta+1} + h_m \bar{b}_m \bar{Z}_m^{\delta+1}). \quad (6.11)$$

To derive the order of stress singularities, we need the expressions of \tilde{u} and \tilde{t} at $\theta = \phi$ and $(\phi \pm \pi)$ where ϕ is a fixed angle. Noticing that Z_m of (6.8b) for $\theta = \phi$ and $(\phi \pm \pi)$ are related by (Ting and Chou 1981A),

$$Z_m(\phi \pm \pi) = e^{\pm i\pi} Z_m(\phi) \quad (6.12)$$

and writing $Z_m(\phi)$ as

$$Z_m(\phi) = r \xi_m(\phi), \quad \xi_m(\phi) = \cos\phi + p_m \sin\phi, \quad (6.13)$$

(6.11) for $\theta = \phi$ and $(\phi \pm \pi)$ become

$$\tilde{t}(\phi) = \sum_{m=1}^3 r^\delta \{ q_m b_m \xi_m^{\delta+1}(\phi) + h_m \bar{b}_m \bar{\xi}_m^{\delta+1}(\phi) \}, \quad (6.14a)$$

$$\tilde{t}(\phi \pm \pi) = \sum_{m=1}^3 r^\delta \{ e^{\pm i(\delta+1)\pi} q_m b_m \xi_m^{\delta+1}(\phi) + e^{\mp i(\delta+1)\pi} h_m \bar{b}_m \bar{\xi}_m^{\delta+1}(\phi) \}. \quad (6.14b)$$

Similar equations can be written for $\tilde{u}(\phi)$ and $\tilde{u}(\phi \pm \pi)$ of (6.8). Introducing the new coefficients

$$\hat{q}_m = q_m \xi_m^{\delta+1}(\phi), \quad (m \text{ not summed}), \quad (6.15a)$$

$$\hat{h}_m = h_m \bar{\xi}_m^{\delta+1}(\phi), \quad (m \text{ not summed}), \quad (6.15b)$$

and noticing that $e^{\pm i(\delta+1)\pi} = -e^{\pm i\delta\pi}$, we have

We may consider following boundary conditions:

$$\underline{\underline{t}}(3\pi/2) = \underline{\underline{0}} \text{ and } \underline{\underline{t}}'(-\pi/2) = \underline{\underline{0}}, \quad (6.19)$$

$$\underline{\underline{t}}(\pi/2) = \underline{\underline{t}}'(\pi/2), \quad (6.20a)$$

$$\underline{u}_1(\pi/2) = \underline{u}'_1(\pi/2), \quad (6.20b)$$

$$t_2 = (\text{sgn } t_2) k_2 t_1 \text{ and } t_3 = (\text{sgn } t_3) k_3 t_1, \text{ at } \theta = \pi/2, \quad (6.20c)$$

where k_2 and k_3 are the coefficients of friction at x_2 and x_3 direction and $(\text{sgn } t_2)$ and $(\text{sgn } t_3)$ stand for the sign of t_2 and t_3 , respectively.

By introducing the matrices

$$\underline{\underline{J}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \underline{\underline{G}} = \begin{bmatrix} 0 & 0 & 0 \\ -(\text{sgn } t_2) k_2 & 1 & 0 \\ -(\text{sgn } t_3) k_3 & 0 & 1 \end{bmatrix}, \quad (6.21)$$

equations (6.20b,c) can be written in matrix notation as

$$\tau^{-1}(\delta+1) \underline{\underline{J}} [\underline{\underline{u}}(\pi/2) - \underline{\underline{u}}'(\pi/2)] + \underline{\underline{G}} \underline{\underline{t}}(\pi/2) = \underline{\underline{0}}. \quad (6.22)$$

Substituting (6.16) and (6.17) into (6.19), (6.20a) and (6.22) and setting $\phi = \pi/2$ lead to

$$e^{i\delta\pi} \underline{\underline{B}} \underline{\underline{q}} + e^{-i\delta\pi} \underline{\underline{B}} \underline{\underline{h}} = \underline{\underline{0}}, \quad (6.23)$$

$$e^{-i\delta\pi} \underline{\underline{B}}' \underline{\underline{q}}' + e^{i\delta\pi} \underline{\underline{B}}' \underline{\underline{h}}' = \underline{\underline{0}}, \quad (6.24)$$

$$\underline{\underline{B}} \underline{\underline{q}} + \underline{\underline{B}} \underline{\underline{h}} = \underline{\underline{B}}' \underline{\underline{q}}' + \underline{\underline{B}}' \underline{\underline{h}}', \quad (6.25)$$

$$\mathbf{J}[(\mathbf{A}\mathbf{q} + \mathbf{B}\mathbf{h}) - (\mathbf{A}'\mathbf{q}' + \mathbf{B}'\mathbf{h}')] + \mathbf{G}[\mathbf{B}\mathbf{q} + \mathbf{B}\mathbf{h}] = \mathbf{0}, \quad (6.26)$$

Equations (6.23)-(6.26) consist of four homogeneous equations for \mathbf{q} , \mathbf{h} , \mathbf{q}' and \mathbf{h}' . For a non-trivial solution the determinant of the coefficient matrix must vanish. This provides the roots for δ . Instead of finding the determinant, we eliminate $\mathbf{B}\mathbf{h}$ and $\mathbf{B}'\mathbf{q}'$ from (6.23), (6.24) and (6.25) to obtain

$$(e^{i\delta\pi} - e^{-i\delta\pi})(\mathbf{B}\mathbf{q} - \mathbf{B}'\mathbf{h}') = \mathbf{0}. \quad (6.27)$$

Hence either

$$(e^{i\delta\pi} - e^{-i\delta\pi}) = 2i \sin\delta\pi = 0, \quad (6.28a)$$

which leads to integer δ or

$$\mathbf{B}\mathbf{q} = \mathbf{B}'\mathbf{h}'. \quad (6.28b)$$

For the latter we substitute \mathbf{h} , \mathbf{h}' and \mathbf{q}' obtained from (6.23), (6.24) and (6.28b) into (6.26). We then have

$$\begin{aligned} & \mathbf{J}[e^{-i\delta\pi}(\mathbf{A}\mathbf{B}^{-1} - \mathbf{A}'\mathbf{B}'^{-1}) - e^{i\delta\pi}(\mathbf{A}\mathbf{B}^{-1} - \mathbf{A}'\mathbf{B}'^{-1})] \\ & - [e^{i\delta\pi} - e^{-i\delta\pi}] \mathbf{G} \mathbf{B}\mathbf{q} = \mathbf{0}. \end{aligned} \quad (6.29)$$

It can be shown (Ting 1986) that

$$\mathbf{A}\mathbf{B}^{-1} - \mathbf{A}'\mathbf{B}'^{-1} = -(\mathbf{W} + i\mathbf{D}), \quad (6.30a)$$

$$\mathbf{A}\mathbf{B}^{-1} - \mathbf{A}'\mathbf{B}'^{-1} = -(\mathbf{W} - i\mathbf{D}), \quad (6.30b)$$

where \underline{W} is real and antisymmetric, while \underline{D} is real, symmetric and positive definite. Equation (6.29) now reduces to

$$\{(\underline{JW} - \underline{G}) - (\cot \delta\pi) \underline{JD}\} \underline{Bq} = \underline{0} . \quad (6.31)$$

For a non-trivial solution of \underline{Bq} , we demand that

$$|(\underline{JW} - \underline{G}) - (\cot \delta\pi) \underline{JD}| = 0 . \quad (6.32)$$

If we expand the determinant, noticing that \underline{W} is antisymmetric and making use of \underline{J} and \underline{G} defined in (6.21), we obtain

$$\cot \delta\pi = \frac{(\operatorname{sgn} t_2)k_2 W_{12} + (\operatorname{sgn} t_3)k_3 W_{13}}{D_{11} + (\operatorname{sgn} t_2)k_2 D_{12} + (\operatorname{sgn} t_3)k_3 D_{13}} . \quad (6.33)$$

We see that when the friction is absent, i.e., $k_2 = k_3 = 0$, $\delta = -1/2$ is the order of singularity.

If we apply the above procedure to the case of transversely isotropic materials under axisymmetric deformation, noticing that $u_3 = t_3 = 0$ in equations (6.19) and (6.20), we obtain the singularity

$$\cot \delta\pi = \frac{(\operatorname{sgn} t_2)k_2 W_{12}}{D_{11} + (\operatorname{sgn} t_2)k_2 D_{12}} . \quad (6.34)$$

For isotropic composites, it can be shown (Ting 1986) that

$$W_{12} = - \left(\frac{1-2\nu'}{\mu'} - \frac{1-2\nu}{\mu} \right) , \quad D_{11} = \frac{1-\nu'}{\mu'} + \frac{1-\nu}{\mu} , \quad (6.35a)$$

$$W_{13} = D_{12} = D_{13} = 0 , \quad (6.35b)$$

where ν and μ are the Poisson's ratio and shear modulus, respectively.

Equation (6.33) then reduces to

$$\cot \delta\pi = (\operatorname{sgn} t_2) k_2 \beta , \quad (6.36a)$$

where β is one of the Dundurs constants (Dundurs 1970)

$$\beta = \frac{\mu(1 - 2\nu') - \mu'(1 - 2\nu)}{\mu(1 - \nu') + \mu'(1 - \nu)} . \quad (6.36b)$$

(II) Case B

We may consider the following boundary conditions:

$$\underline{\underline{t}}(\pi/2) = \underline{\underline{t}}'(\pi/2) \quad \text{and} \quad \underline{\underline{u}}(\pi/2) = \underline{\underline{u}}'(\pi/2) , \quad (6.37)$$

$$\underline{\underline{t}}(3\pi/2) = \underline{\underline{t}}'(-\pi/2) , \quad (6.38a)$$

$$\underline{\underline{u}}_1(3\pi/2) = \underline{\underline{u}}_1'(-\pi/2) , \quad (6.38b)$$

$$t_2 = (\operatorname{sgn} t_2) k_2 t_1 \text{ and } t_3 = (\operatorname{sgn} t_3) k_3 t_1 , \quad \text{at } \theta = 3\pi/2 . \quad (6.38c)$$

Equations (6.38b,c) can be written in matrix notation as

$$\underline{\underline{r}}^{-1}(\delta+1) \underline{\underline{J}} [\underline{\underline{u}}(3\pi/2) - \underline{\underline{u}}'(-\pi/2)] + \underline{\underline{G}} \underline{\underline{t}}(3\pi/2) = \underline{\underline{0}} , \quad (6.39)$$

where k_2 , k_3 , $\operatorname{sgn} t_2$, $\operatorname{sgn} t_3$, $\underline{\underline{J}}$ and $\underline{\underline{G}}$ are the same as in Case A. Substituting (6.16) and (6.17) into (6.37), (6.38a) and (6.39) and setting $\phi = \pi/2$ lead to

$$\underline{\underline{B}} \underline{\underline{q}} + \underline{\underline{\bar{B}}} \underline{\underline{h}} = \underline{\underline{B}}' \underline{\underline{q}}' + \underline{\underline{\bar{B}}}' \underline{\underline{h}}' , \quad (6.40)$$

$$\underline{\underline{A}}\underline{\underline{q}} + \underline{\underline{A}}\underline{\underline{h}} = \underline{\underline{A}}'\underline{\underline{q}}' + \underline{\underline{A}}'\underline{\underline{h}}', \quad (6.41)$$

$$e^{i\delta\pi}\underline{\underline{B}}\underline{\underline{q}} + e^{-i\delta\pi}\underline{\underline{B}}\underline{\underline{h}} = e^{-i\delta\pi}\underline{\underline{B}}'\underline{\underline{q}}' + e^{i\delta\pi}\underline{\underline{B}}'\underline{\underline{h}}', \quad (6.42)$$

$$J[-(e^{i\delta\pi}\underline{\underline{A}}\underline{\underline{q}} + e^{-i\delta\pi}\underline{\underline{A}}\underline{\underline{h}}) + (e^{-i\delta\pi}\underline{\underline{A}}'\underline{\underline{q}}' + e^{i\delta\pi}\underline{\underline{A}}'\underline{\underline{h}}')], \quad (6.43)$$

$$-G[e^{i\delta\pi}\underline{\underline{B}}\underline{\underline{q}} + e^{-i\delta\pi}\underline{\underline{B}}\underline{\underline{h}}] = 0.$$

Equations (6.40)-(6.43) consist of four homogeneous equations for $\underline{\underline{q}}$, $\underline{\underline{h}}$, $\underline{\underline{q}}'$ and $\underline{\underline{h}}'$. For a non-trivial solution, the determinant of the coefficient matrix must vanish. This provides the roots for δ . Once again, by algebraic operations, we obtain that δ is an integer or the root of following determinant

$$\begin{vmatrix} \underline{\underline{D}} & \underline{\underline{W}} \\ \underline{\underline{JW}} - G & -(\cot \delta\pi) \underline{\underline{G}} - \underline{\underline{JD}} \end{vmatrix} = 0, \quad (6.44)$$

where $\underline{\underline{D}}$ and $\underline{\underline{W}}$ are the same as in Case A.

If we apply the above procedure to the case of transversely isotropic materials under axisymmetric deformation, noticing that $u_3 = t_3 = 0$ in equations (6.37) and (6.38), we obtain the singularity

$$\cot \delta\pi = \frac{-(\operatorname{sgn} t_2) k_2 W_{12}}{D_{11} + (\operatorname{sgn} t_2) k_2 D_{12}}. \quad (6.45)$$

When the friction is absent, i.e., $k_2 = 0$, $\delta = -1/2$ is the order of singularity.

For isotropic composites (6.44) reduces to

$$\cot \delta\pi = -(\operatorname{sgn} t_2) k_2 \beta, \quad (6.46)$$

where β is one of the Dundurs constants given in (6.36b). Equation (6.46) agrees with the result obtained by Comninou (1978B).

We have verified that δ obtained from (6.34) and (6.45) agreed with that obtained from (4.12) for Case A and B, respectively. Since δ is real, the unrealistic inter-penetration of the crack surfaces does not exist.

Chapter VII
CONCLUDING REMARKS

The problem of stress singularity in a three-dimensional elastic solid that contains axisymmetric notches or cracks and subjected to an axisymmetric deformation has been reduced to a mathematically two-dimensional problem. In this case, it has been shown that the eigenfunctions for the singularity associated with an eigenvalue δ contain not only the term $\rho^\delta f(\psi, \delta)$, but also the terms $\rho^{\delta+1} f_1(\psi, \delta)$, $\rho^{\delta+2} f_2(\psi, \delta)$... where (ρ, ψ) is the polar coordinate with origin at the apex of notches or cracks. In the case of interface crack with a contact zone, it can be seen from (6.34) and (6.45) that if δ is an eigenvalue, so is $\delta+k$ where k is an integer. For the high order terms of the eigenfunction solution, equations (4.7b) must be solved. Numerical calculation shows that (4.7b) has no solution for $k = 1$. To obtain the high order terms for $k \geq 1$ the modified solution of (3.8), which is obtained by differentiating (2.27) with respect to δ , has been used. A solution for term $k = 1$ is thus obtained but (3.8c) and (3.8d) have no numerical solution for $k = 2$. To obtain the terms associate with $k > 2$ one has to find the new solutions by taking second or higher derivatives with respect to δ .

From the numerical computations we present the following conclusions

(1) It is shown in Chapter VI that the first term in the eigenfuncion series solution of axisymmetric deformation is the same as the solution of plane strain problem. The singularity δ of two isotropic materials obtained by the formulas here agrees with the results of Bogy(1971) and Lin and Mar (1976).

(2) When the material constants of two transversely isotropic materials in a composite are chosen in such a way that they are very close to two isotropic materials, the order of singularity obtained by the formulas of transversely isotropic composite and of isotropic composite are very close.

(3) The singularities obtained by the methods of Chapter V and Chapter VI are exactly same.

Appendix A

DERIVATION OF EQUATIONS (2.13)

Substitution of (2.9a) into (2.8) and noticing that $\phi^{(k)}$ is of order $\rho^{\delta+k+2}$, we see that (2.8) is satisfied if

$$\frac{\partial^2 \phi^{(0)}}{\partial x^2} - \frac{1}{p^2} \frac{\partial^2 \phi^{(0)}}{\partial z^2} = 0, \quad (A1)$$

$$\frac{\partial^2 \phi^{(k)}}{\partial x^2} - \frac{1}{p^2} \frac{\partial^2 \phi^{(k)}}{\partial z^2} = \sum_{m=0}^{k-1} x^{k-1-m} \frac{\partial \phi^{(m)}}{\partial x}, \quad k \geq 1. \quad (A2)$$

Thus each term in (A2) is of order $\rho^{\delta+k}$. Using (2.9b) in (A2) we obtain

$$\begin{aligned} & \sum_{t=0}^{k-1} A_{t+1}^{(k)} t(t+1) x^{t-1} z^{\delta+k-t+1} + \sum_{t=1}^k A_t^{(k)} 2tx^{t-1} z^{\delta+k-t+1} (\delta+k-t+2) \\ & = J_1 + J_2, \end{aligned} \quad (A3)$$

where

$$J_1 = \sum_{m=1}^{k-1} x^{k-1-m} \sum_{s=0}^{m-1} A_{s+1}^{(m)} (s+1) x^s z^{\delta+m-s+1}, \quad (A4)$$

$$J_2 = \sum_{m=1}^{k-1} x^{k-1-m} \sum_{s=0}^m A_s^{(m)} x^s z^{\delta+m-s+1} (\delta+m-s+2). \quad (A5)$$

By letting $s = t+m-k$ and interchanging the summations, we have

$$J_1 = \sum_{t=1}^{k-1} \sum_{m=k-t}^{k-1} (t-k+m+1) A_{t-k+m+1}^{(m)} x^{t-1} z^{\delta+k-t+1}, \quad (A6)$$

$$J_2 = \sum_{t=1}^{k-1} \sum_{m=k-t}^{k-1} A_{t-k+m}^{(m)} x^{t-1} z^{\delta+k-t+1} (\delta+k-t+2). \quad (A7)$$

By setting the coefficients of the same terms in (A3) to zero, we obtain

$$2kA_k^{(k)} = \sum_{m=0}^{k-1} A_m^{(m)}, \quad (A8)$$

$$\begin{aligned} t(t+1)A_{t+1}^{(k)} + 2t(\delta+k-t+2)A_t^{(k)} \\ = \sum_{m=k-t}^{k-1} \{(t-k+m+1)A_{t-k+m+1}^{(m)} + (\delta+k-t+2)A_{t-k+m}^{(m)}\}. \end{aligned} \quad (A9)$$

Equation (A9) can be rewritten as, by letting $m = s+k-t-1$,

$$\begin{aligned} t(t+1)A_{t+1}^{(k)} + 2t(\delta+k-t+2)A_t^{(k)} \\ = \sum_{s=1}^t \{sA_s^{s+k-t-1} + (\delta+k-t+2)A_{s-1}^{s+k-t-1}\}, \quad \text{for } t \leq k-1. \end{aligned} \quad (A10)$$

To express (A8) in the form of (2.13a), we replace k by $k-1$ in (A8) to obtain

$$2(k-1)A_{k-1}^{(k-1)} = \sum_{m=0}^{k-2} A_m^{(m)}, \quad (A11)$$

and subtract (A11) from (A8). We have

$$2kA_k^{(k)} - 2(k-1)A_{k-1}^{(k-1)} = A_{k-1}^{(k-1)}, \quad (A12)$$

which is identical to (2.13a). Similarly, we replace k and t by $k-1$ and $t-1$, respectively, in (A10) which reduces to

$$\begin{aligned} (t-1)tA_t^{(k-1)} + 2(t-1)(\delta+k-t+2)A_{t-1}^{(k-1)} \\ = \sum_{s=1}^{t-1} \{sA_s^{s+k-t-1} + (\delta+k-t+2)A_{s-1}^{s+k-t-1}\}. \end{aligned} \quad (A13)$$

Equation (2.13b) is obtained when (A13) is subtracted from (A10).

Appendix B

MODIFIED SOLUTIONS IN SECTION (3.1)

We obtain, by differentiating equation (2.14a), (2.17) and (2.18) with respect to δ , the following modified solutions

$$\phi^{(k)} = \frac{\partial}{\partial \delta} \left\{ \sum_{t=0}^k A_t^{(k)} x^t z^{\delta+k-t+2} + \dots \right\}, \quad (B1)$$

$$\begin{aligned} u_r^{(k)} = & \sum_{t=0}^k \left\{ [A_t^{(k)} (\delta+k-t+2) + A_{t+1}^{(k)} (t+1) + A_t^{(k)}] x^t z^{\delta+k-t+1} \right. \\ & \left. + [A_t^{(k)} (\delta+k-t+2) + A_{t+1}^{(k)} (t+1)] x^t z^{\delta+k-t+1} \ln z \right\} + \dots \end{aligned} \quad (B2)$$

$$\begin{aligned} u_z^{(k)} = & \sum_{t=0}^k \left\{ [A_t^{(k)} m_{,p_1} (\delta+k-t+2) + A_t^{(k)} m_{,p_1}] x^t z^{\delta+k-t+1} \right. \\ & \left. + A_t^{(k)} m_{,p_1} (\delta+k-t+2) x^t z^{\delta+k-t+1} \ln z \right\} + \dots \end{aligned} \quad (B3)$$

$$\begin{aligned} \sigma_r^{(k)} = & \sum_{t=0}^k \left[-A_t^{(k)} c_{44} (1+m_{,p_1}) p_1^2 (\delta+k-t+2) (\delta+k-t+1) \right. \\ & + A_{t+1}^{(k)} 2(c_{11} - c_{12}) (t+1) (\delta+k-t+1) \\ & + A_{t+2}^{(k)} (c_{11} - c_{12}) (t+2) (t+1) \\ & - A_t^{(k)} c_{44} (1+m_{,p_1}) p_1^2 (2\delta+2k-2t+3) \\ & + A_{t+1}^{(k)} 2(c_{11} - c_{12}) (t+1)] x^t z^{\delta+k-t} \\ & + \left. [-A_t^{(k)} c_{44} (1+m_{,p_1}) p_1^2 (\delta+k-t+2) (\delta+k-t+1) \right. \\ & + A_{t+1}^{(k)} 2(c_{11} - c_{12}) (t+1) (\delta+k-t+1) \\ & \left. + A_{t+2}^{(k)} (c_{11} - c_{12}) (t+2) (t+1)] x^t z^{\delta+k-t} \ln z \right\} + \dots \end{aligned} \quad (B4)$$

$$\begin{aligned}
\sigma_{\theta}^{(k)} = & \sum_{t=0}^k [A_t^{(k)} (c_{1,2} + c_{1,3} m_1 p_1^2) (\delta+k-t+2) (\delta+k-t+1) \\
& - A_{t+1}^{(k)} 2(c_{1,1} - c_{1,2}) (t+1) (\delta+k-t+1) \\
& - A_{t+2}^{(k)} (t+2) (t+1) \\
& + A_t^{(k)} (c_{1,2} + c_{1,3} m_1 p_1^2) (2\delta+2k-2t+3) \\
& - A_{t+1}^{(k)} 2(c_{1,1} - c_{1,2}) (t+1)] x^t z_1^{\delta+k-t} \\
& + [A_t^{(k)} (c_{1,2} + c_{1,3} m_1 p_1^2) (\delta+k-t+2) (\delta+k-t+1) \\
& - A_{t+1}^{(k)} 2(c_{1,1} - c_{1,2}) (t+1) (\delta+k-t+1) \\
& - A_{t+2}^{(k)} (t+2) (t+1)] x^t z_1^{\delta+k-t} \ln z_1 \} + \dots
\end{aligned} \tag{B5}$$

$$\begin{aligned}
\sigma_z^{(k)} = & \sum_{t=0}^k \{ [-A_t^{(k)} c_{4,4} (1+m_1) (\delta+k-t+2) (\delta+k-t+1) \\
& - A_t^{(k)} c_{4,4} (1+m_1) (2\delta+2k-2t+3)] x^t z_1^{\delta+k-t} \\
& [-A_t^{(k)} c_{4,4} (1+m_1) (\delta+k-t+2) (\delta+k-t+1)] x^t z_1^{\delta+k-t} \ln z_1 \} + \dots
\end{aligned} \tag{B6}$$

$$\begin{aligned}
\sigma_{xz}^{(k)} = & \sum_{t=0}^k \{ [A_t^{(k)} (\delta+k-t+2) (\delta+k-t+1) + A_{t+1}^{(k)} (t+1) (\delta+k-t+1) \\
& + A_t^{(k)} (2\delta+2k-2t+3) + A_{t+1}^{(k)} (t+1)] c_{4,4} (1+m_1) p_1 x^t z_1^{\delta+k-t} \\
& [A_t^{(k)} (\delta+k-t+2) + A_{t+1}^{(k)} (t+1)] c_{4,4} (1+m_1) p_1 (\delta+k-t+1) x^t z_1^{\delta+k-t} \ln z_1 \} + \dots
\end{aligned} \tag{B7}$$

For the degenerate materials, we obtain from equations (2.37), (2.38) and (2.39)

$$\begin{aligned}
\phi^{(k)} = & \frac{\partial}{\partial \delta} \{ \sum_{t=0}^k [A_t^{(k)} x^t z^{\delta+k-t+2} + C_t^{(k)} x^t z^{\delta+k-t+2}] \\
& + \sum_{t=0}^k [B_t^{(k)} z x^t z^{\delta+k-t+1} + D_t^{(k)} z x^t z^{\delta+k-t+1}] (\delta+k-t+2) \} ,
\end{aligned} \tag{B8}$$

$$u_r^{(k)} = \sum_{t=0}^k \{ [A_t^{(k)} \delta + k - t + 2 + A_{t+1}^{(k)} (t+1) + A_t^{(k)}] x^t z^{\delta + k - t + 1} \\ + [B_t^{(k)} \delta + k - t + 2 (\delta + k - t + 1) + B_{t+1}^{(k)} (t+1) (\delta + k - t + 1) \\ + B_t^{(k)} (2\delta + 2k - 2t + 3) + B_{t+1}^{(k)} (t+1)] zx^t z^{\delta + k - t} \\ + [A_t^{(k)} \delta + k - t + 2 + A_{t+1}^{(k)} (t+1)] x^t z^{\delta + k - t + 1} \ln z \\ + [B_t^{(k)} \delta + k - t + 2 + B_{t+1}^{(k)} (t+1)] (\delta + k - t + 1) zx^t z^{\delta + k - t} \ln z \} + \dots$$
(B9)

$$u_z^{(k)} = \sum_{t=0}^k \{ [A_t^{(k)} \beta_i (\delta + k - t + 2) - B_t^{(k)} (3 - 4\nu) (\delta + k - t + 2) \\ + A_t^{(k)} \beta_i - B_t^{(k)} (3 - 4\nu)] x^t z^{\delta + k - t + 1} \\ + [B_t^{(k)} \beta_i (\delta + k - t + 2) (\delta + k - t + 1) \\ + B_t^{(k)} \beta_i (2\delta + 2k - 2t + 3)] zx^t z^{\delta + k - t} \\ + [A_t^{(k)} \beta_i - B_t^{(k)} (3 - 4\nu)] (\delta + k - t + 2) x^t z^{\delta + k - t + 1} \ln z \\ + B_t^{(k)} \beta_i (\delta + k - t + 2) (\delta + k - t + 1) zx^t z^{\delta + k - t} \ln z \} + \dots$$
(B10)

$$\frac{1}{2\mu} \sigma_r^{(k)} = \sum_{t=0}^k \{ [(A_t^{(k)} \beta^2 - B_t^{(k)} 2\nu \beta_i) (\delta + k - t + 2) (\delta + k - t + 1) \\ + A_{t+1}^{(k)} 2\gamma(t+1) (\delta + k - t + 1) + A_{t+2}^{(k)} \gamma(t+2) (t+1)] x^t z^{\delta + k - t} \\ + [B_t^{(k)} \beta^2 (\delta + k - t + 2) (\delta + k - t + 1) + B_{t+1}^{(k)} 2\gamma(t+1) (\delta + k - t + 1) \\ + B_{t+2}^{(k)} \gamma(t+2) (t+1)] (\delta + k - t) zx^t z^{\delta + k - t - 1} \\ + [(A_t^{(k)} \beta^2 - B_t^{(k)} 2\nu \beta_i) (2\delta + 2k - 2t + 3) + A_{t+1}^{(k)} 2\gamma(t+1)] x^t z^{\delta + k - t} \\ + [B_t^{(k)} \beta^2 ((2\delta + 2k - 2t + 3) (\delta + k - t) + (\delta + k - t + 2) (\delta + k - t + 1)) \\ + B_{t+1}^{(k)} 2\gamma(t+1) (2\delta + 2k - 2t + 1) + B_{t+2}^{(k)} \gamma(t+2) (t+1)] zx^t z^{\delta + k - t - 1} \\ + [(A_t^{(k)} \beta^2 - B_t^{(k)} 2\nu \beta_i) (\delta + k - t + 2) (\delta + k - t + 1) \\ + A_{t+1}^{(k)} 2\gamma(t+1) (\delta + k - t + 1) + A_{t+2}^{(k)} \gamma(t+2) (t+1)] x^t z^{\delta + k - t} \ln z \\ + [B_t^{(k)} \beta^2 (\delta + k - t + 2) (\delta + k - t + 1) + B_{t+1}^{(k)} 2\gamma(t+1) (\delta + k - t + 1) \\ + B_{t+2}^{(k)} \gamma(t+2) (t+1)] (\delta + k - t) zx^t z^{\delta + k - t - 1} \ln z \} + \dots$$
(B11)

$$\begin{aligned}
 \frac{1}{2\mu} \sigma_{\theta}^{(k)} = \sum_{t=0}^k & \{ [(A_t^{(k)} (\beta^2 - \gamma) - B_t^{(k)} 2\nu \beta i) (\delta + k - t + 2) (\delta + k - t + 1) \\
 & - A_{t+1}^{(k)} 2\gamma(t+1) (\delta + k - t + 1) - A_{t+2}^{(k)} \gamma(t+2) (t+1)] x^t z^{\delta+k-t} \\
 & + [B_t^{(k)} (\beta^2 - \gamma) (\delta + k - t + 2) (\delta + k - t + 1) - B_{t+1}^{(k)} 2\gamma(t+1) (\delta + k - t + 1) \\
 & - B_{t+2}^{(k)} \gamma(t+2) (t+1)] (\delta + k - t) zx^t z^{\delta+k-t-1} \\
 & + [(A_t^{(k)} (\beta^2 - \gamma) - B_t^{(k)} 2\nu \beta i) (2\delta + 2k - 2t + 3) - A_{t+1}^{(k)} 2\gamma(t+1)] x^t z^{\delta+k-t} \\
 & + [B_t^{(k)} (\beta^2 - \gamma) ((\delta + k - t + 2) (\delta + k - t + 1) + (2\delta + 2k - 2t + 3)) \\
 & - B_{t+1}^{(k)} 2\gamma(t+1) (2\delta + 2k - 2t + 1) - B_{t+2}^{(k)} \gamma(t+2) (t+1)] zx^t z^{\delta+k-t-1} \\
 & + [(A_t^{(k)} (\beta^2 - \gamma) - B_t^{(k)} 2\nu \beta i) (\delta + k - t + 2) (\delta + k - t + 1) \\
 & - A_{t+1}^{(k)} 2\gamma(t+1) (\delta + k - t + 1) - A_{t+2}^{(k)} \gamma(t+2) (t+1)] x^t z^{\delta+k-t} \ln z \\
 & + [B_t^{(k)} (\beta^2 - \gamma) (\delta + k - t + 2) (\delta + k - t + 1) - B_{t+1}^{(k)} 2\gamma(t+1) (\delta + k - t + 1) \\
 & - B_{t+2}^{(k)} \gamma(t+2) (t+1)] (\delta + k - t) zx^t z^{\delta+k-t-1} \ln z \} + \dots
 \end{aligned} \tag{B12}$$

$$\begin{aligned}
 \frac{1}{2\mu} \sigma_z^{(k)} = \sum_{t=0}^k & \{ - [A_t^{(k)} + B_t^{(k)} 2(1-\nu) i \beta^{-1}] (\delta + k - t + 2) (\delta + k - t + 1) x^t z^{\delta+k-t} \\
 & - B_t^{(k)} (\delta + k - t + 2) (\delta + k - t + 1) (\delta + k - t) zx^t z^{\delta+k-t-1} \\
 & - [A_t^{(k)} + B_t^{(k)} 2(1-\nu) i \beta^{-1}] (2\delta + 2k - 2t + 3) x^t z^{\delta+k-t} \\
 & - B_t^{(k)} (\delta + k - t + 2) (\delta + k - t + 1) + (2\delta + 2k - 2t + 3) (\delta + k - t)] zx^t z^{\delta+k-t-1} \\
 & - [A_t^{(k)} + B_t^{(k)} 2(1-\nu) i \beta^{-1}] (\delta + k - t + 2) (\delta + k - t + 1) x^t z^{\delta+k-t} \ln z \\
 & - B_t^{(k)} (\delta + k - t + 2) (\delta + k - t + 1) (\delta + k - t) zx^t z^{\delta+k-t-1} \ln z \} + \dots
 \end{aligned} \tag{B13}$$

$$\begin{aligned}
 \frac{1}{2\mu} \sigma_{xz}^{(k)} = & \sum_{t=0}^k \{ [(A_t^{(k)} B_{i-t} - B_t^{(k)} (1-2\nu)) (\delta+k-t+2) \\
 & + (A_{t+1}^{(k)} B_{i-t} - B_{t+1}^{(k)} (1-2\nu)) (t+1)] (\delta+k-t+1) x^t z^{\delta+k-t} \\
 & + [B_t^{(k)} (\delta+k-t+2) + B_{t+1}^{(k)} (t+1)] (\delta+k-t+1) (\delta+k-t) \beta_{izx} t z^{\delta+k-t-1} \\
 & [(A_t^{(k)} B_{i-t} - B_t^{(k)} (1-2\nu)) (2\delta+2k-2t+3) \\
 & + (A_{t+1}^{(k)} B_{i-t} - B_{t+1}^{(k)} (1-2\nu)) (t+1)] x^t z^{\delta+k-t} \\
 & + [B_t^{(k)} (\delta+k-t+2) (\delta+k-t+1) + (2\delta+2k-2t+3) (\delta+k-t)] \\
 & + B_{t+1}^{(k)} (t+1) (2\delta+2k-2t+3)] \beta_{izx} t z^{\delta+k-t-1} \\
 & + [(A_t^{(k)} B_{i-t} - B_t^{(k)} (1-2\nu)) (\delta+k-t+2) \\
 & + (A_{t+1}^{(k)} B_{i-t} - B_{t+1}^{(k)} (1-2\nu)) (t+1)] (\delta+k-t+1) x^t z^{\delta+k-t-1} \ln z \\
 & + [B_t^{(k)} (\delta+k-t+2) + B_{t+1}^{(k)} (t+1)] (\delta+k-t+1) (\delta+k-t) \beta_{izx} t z^{\delta+k-t-1} \ln z \} + \dots
 \end{aligned} \tag{B14}$$

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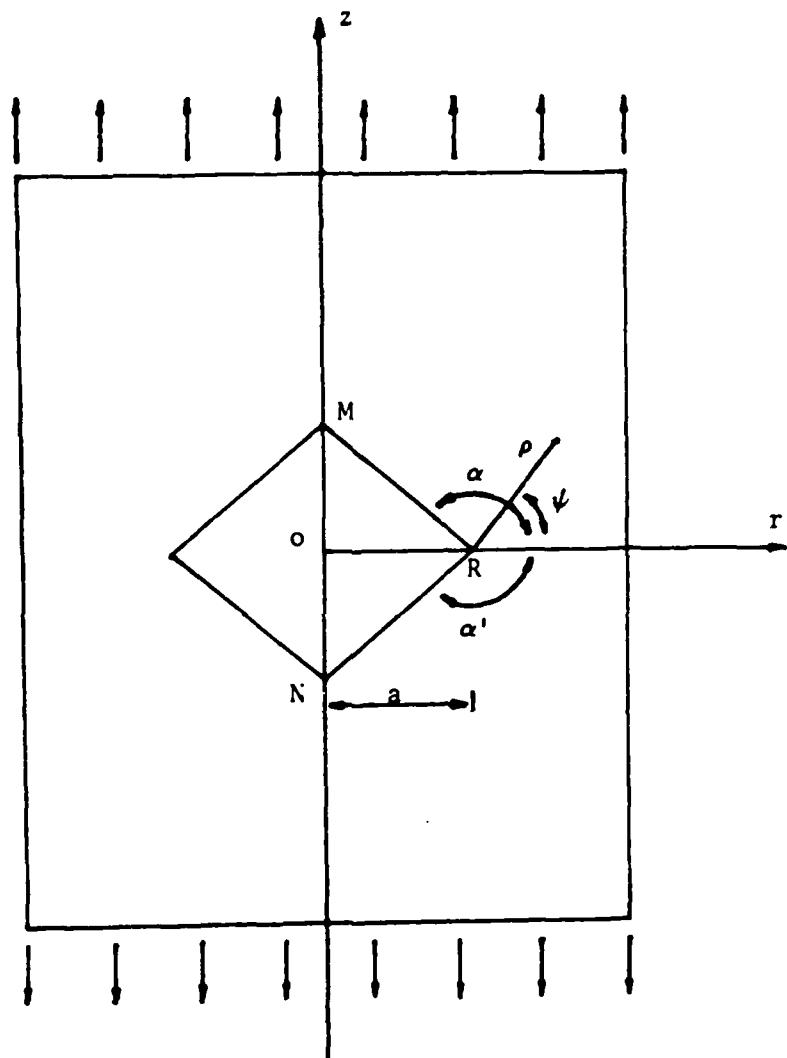


Fig.1 Cross section of an axisymmetric body that contains notches

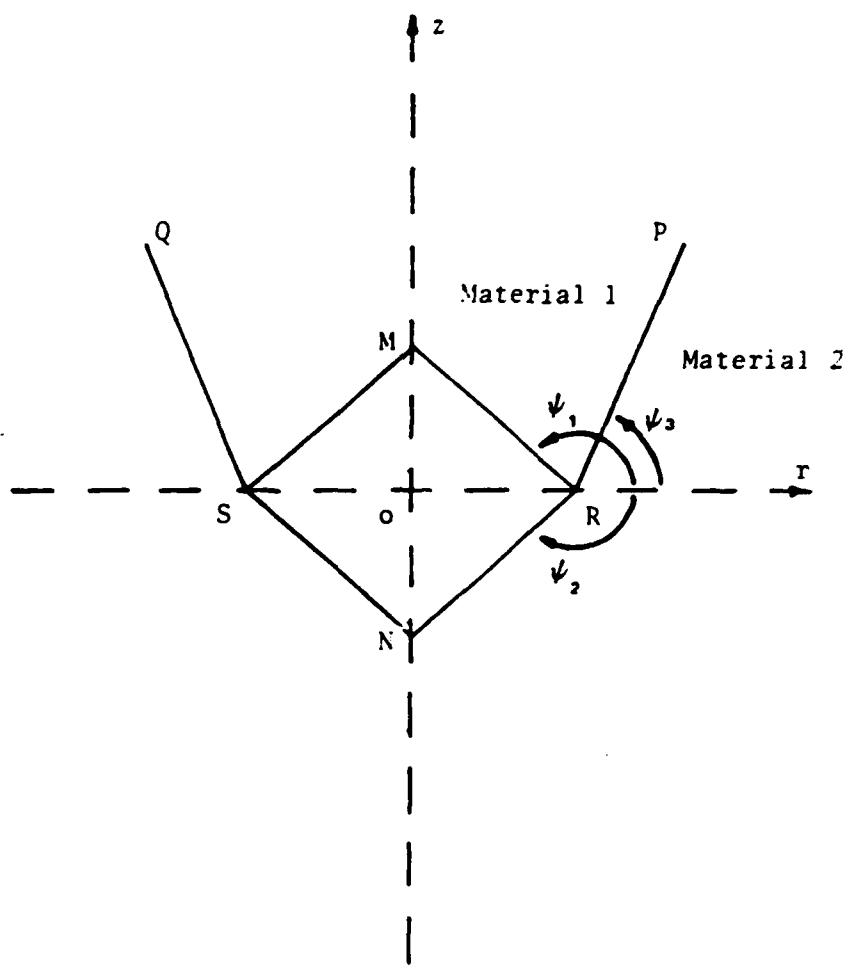


Fig.2 Cross section of an axisymmetric composite that contains notches

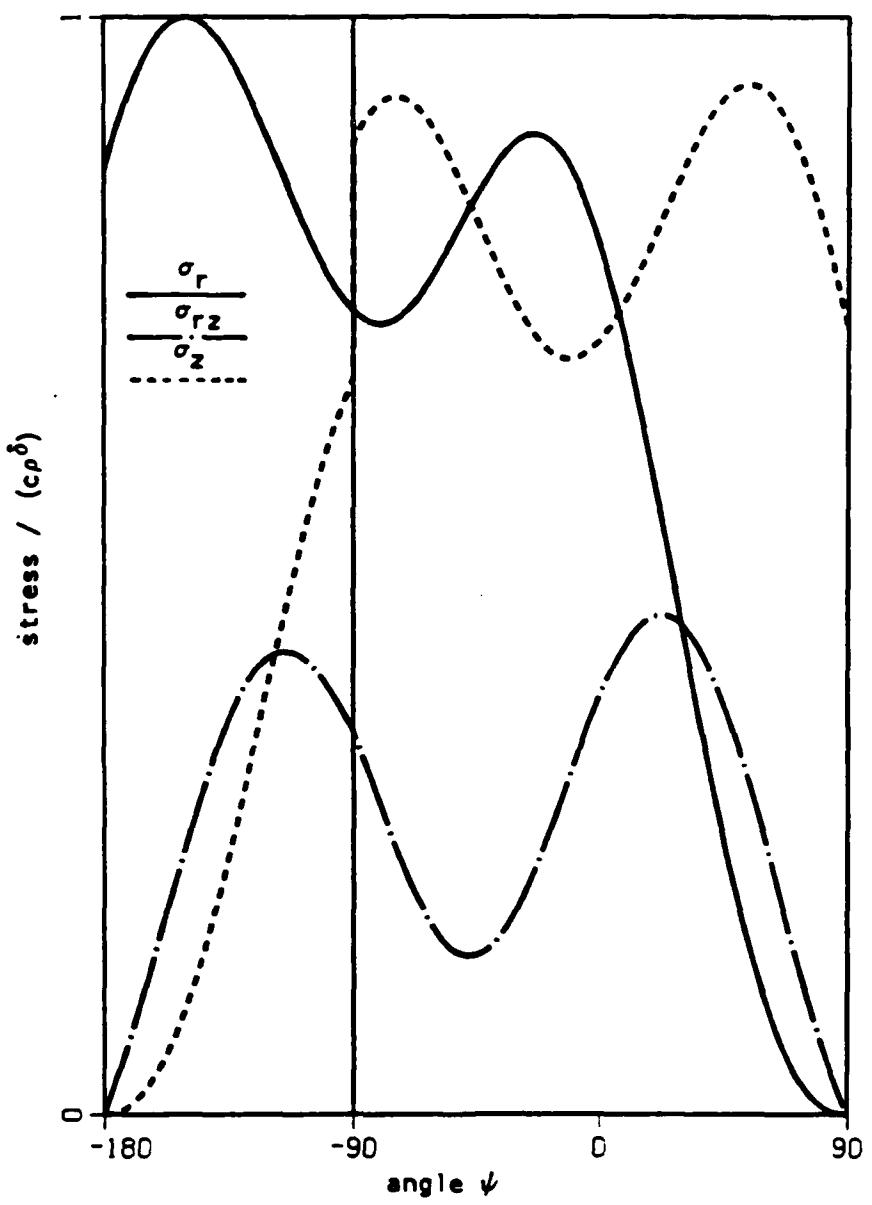


Fig.3 The normalized stresses from the first term of the eigenfunction associated with δ_1 of Example 1

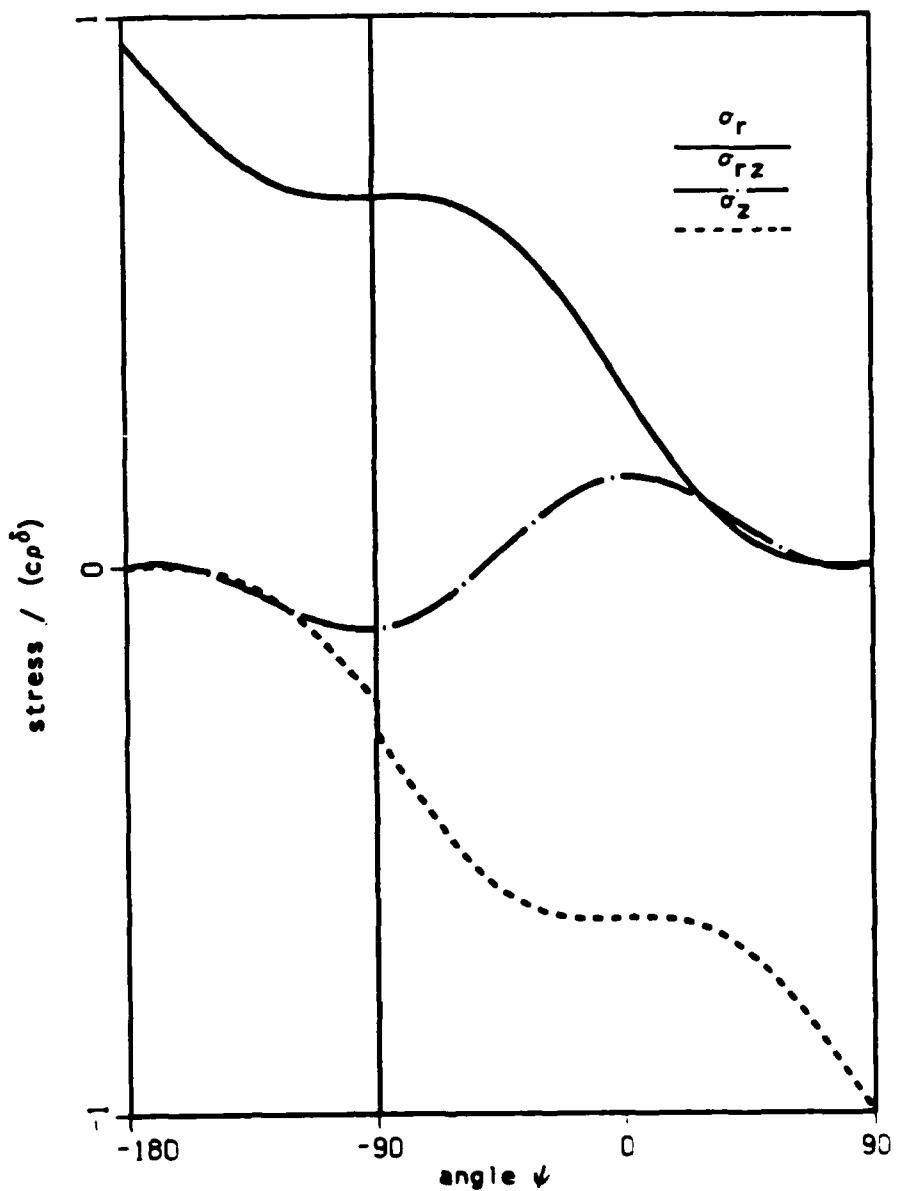


Fig.4 The normalized stresses from the first term of the eigenfunction associated with δ_1 of Example 1

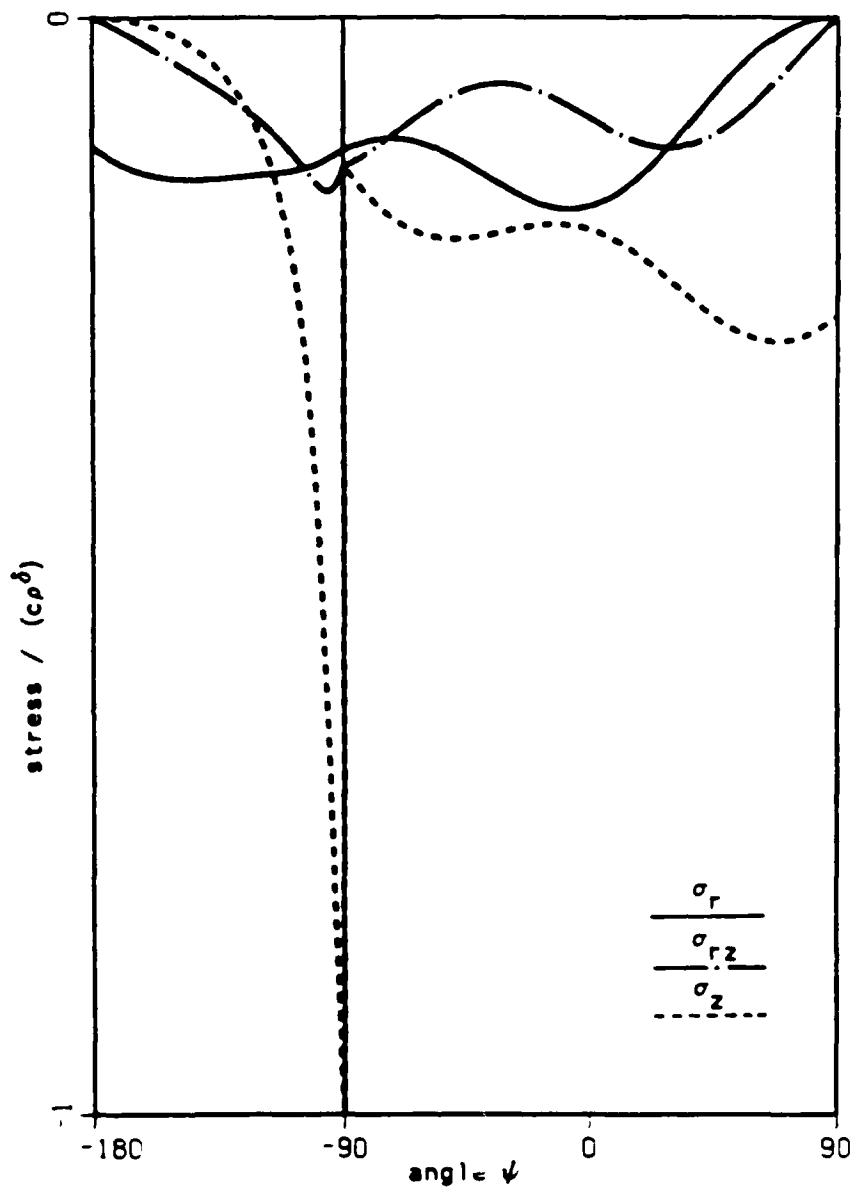


Fig.5 The normalized stresses from the first term of the eigenfunction associated with δ_1 of Example 2

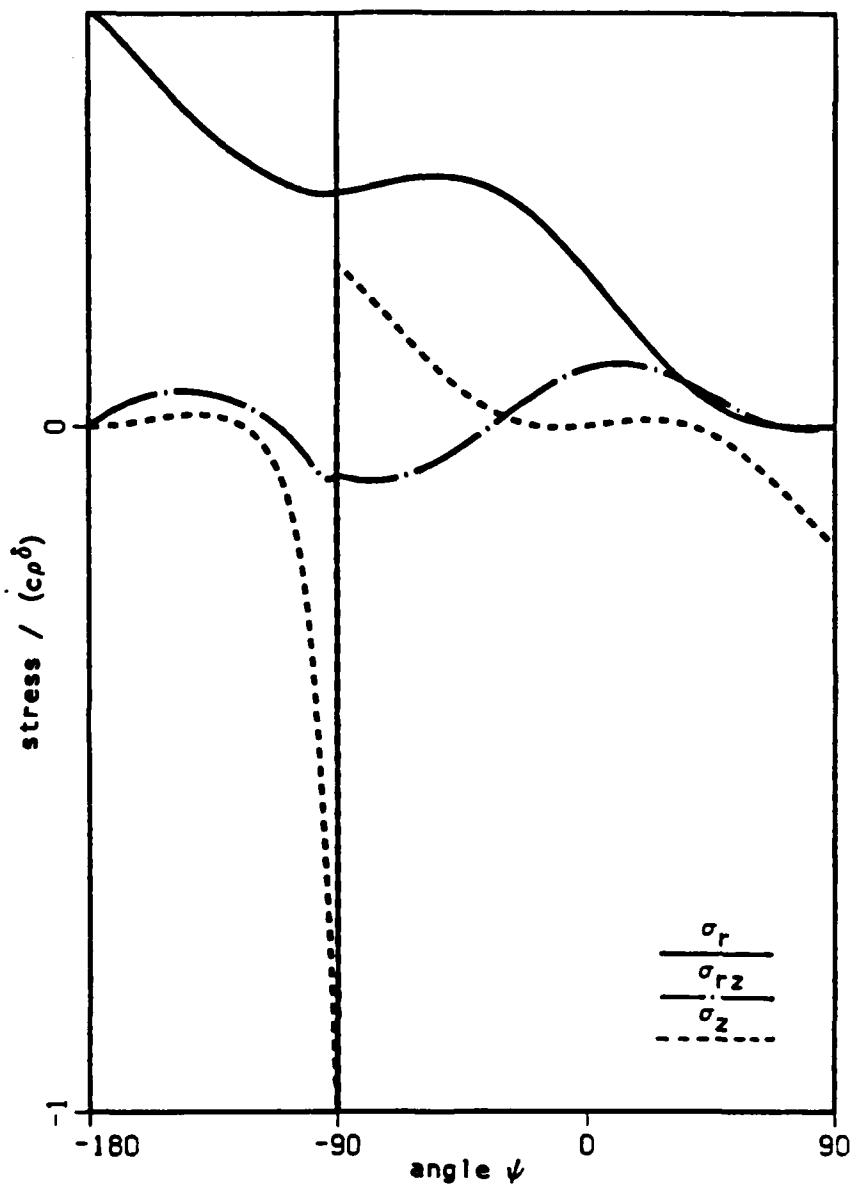


Fig.6 The normalized stresses from the first term of the eigenfunction associated with δ_2 of Example 2

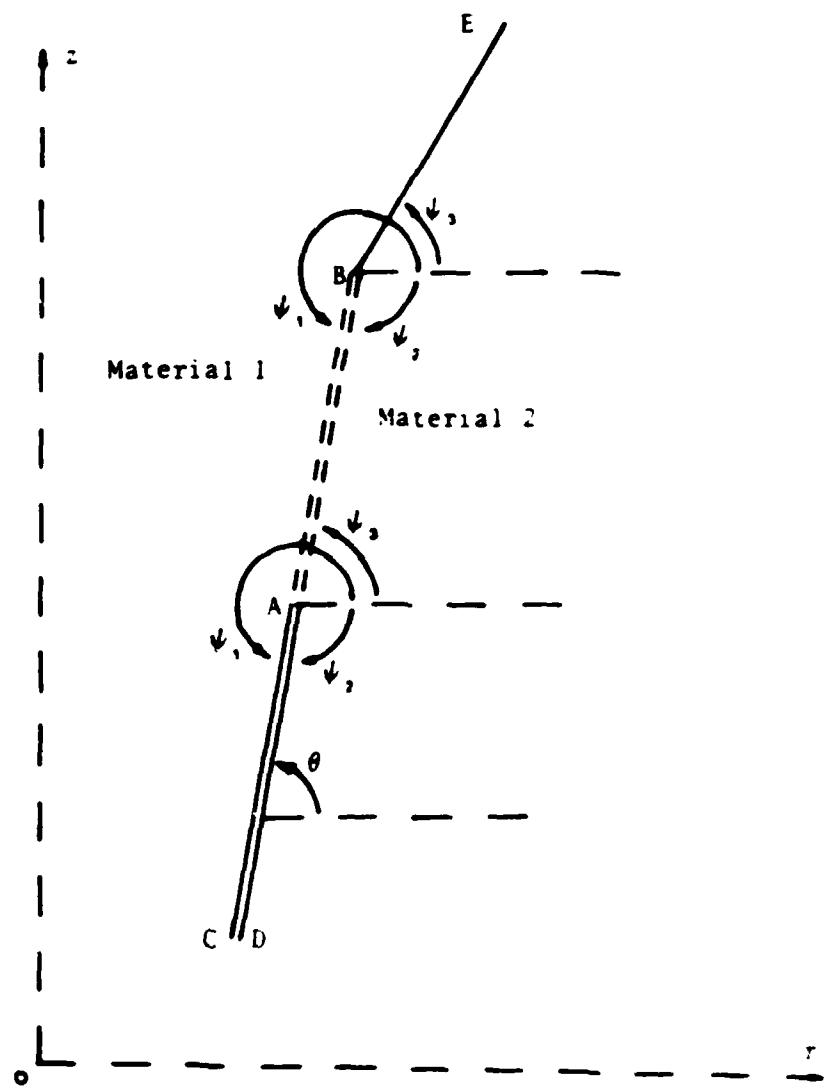


Fig.7 Cross section of an axisymmetric composite that contains an interface crack with a contact zone

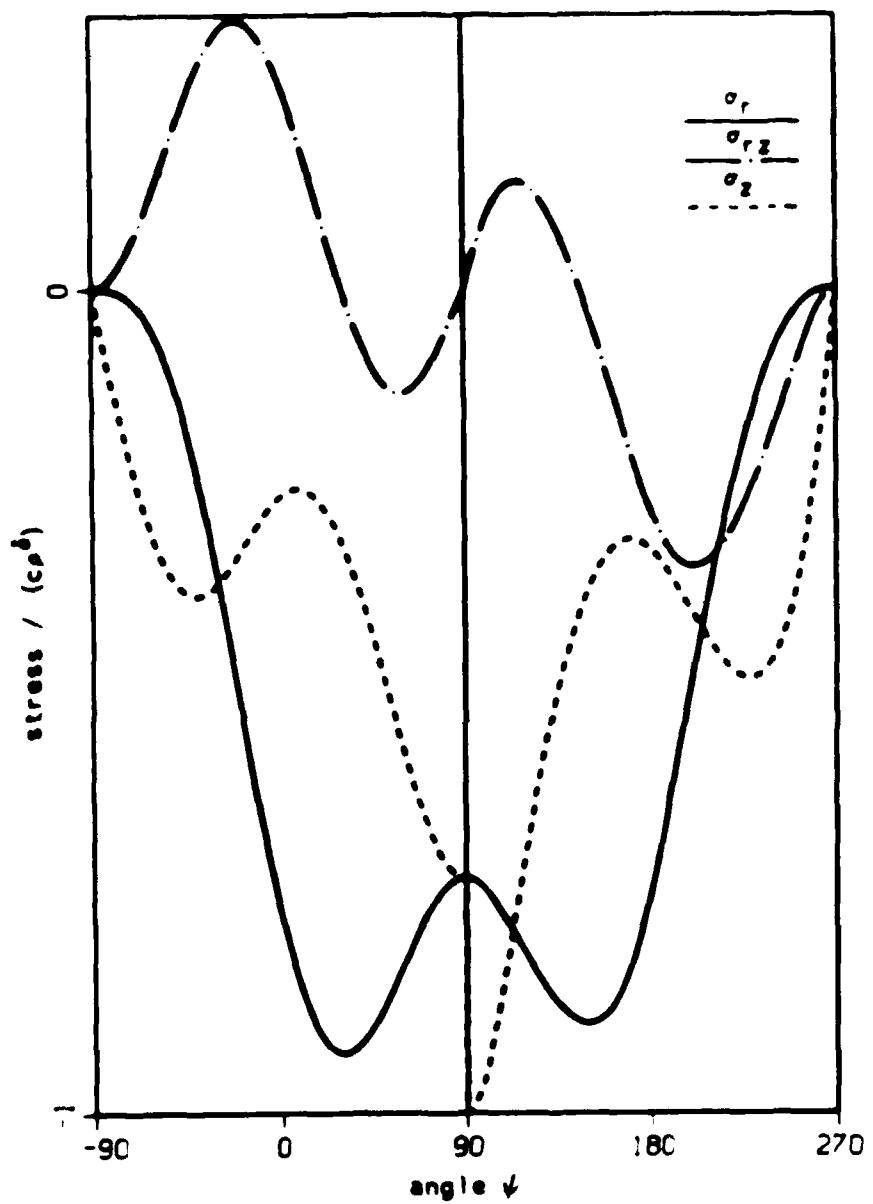


Fig.8 The normalized stresses from the first term of the eigenfunction
in Case A of an interface crack with frictionless contact

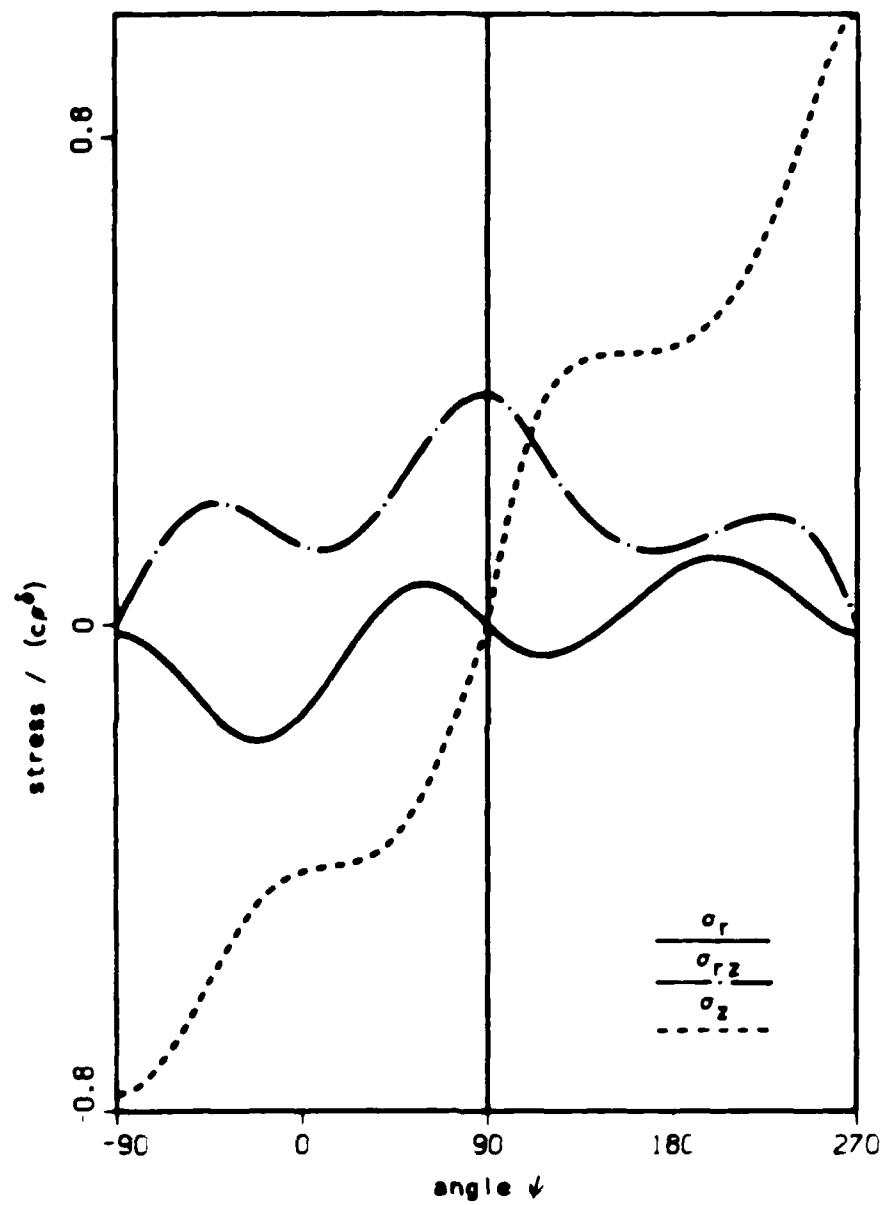


Fig.9 The normalized stresses from the first term of the eigenfunction in Case B of an interface crack with frictionless contact

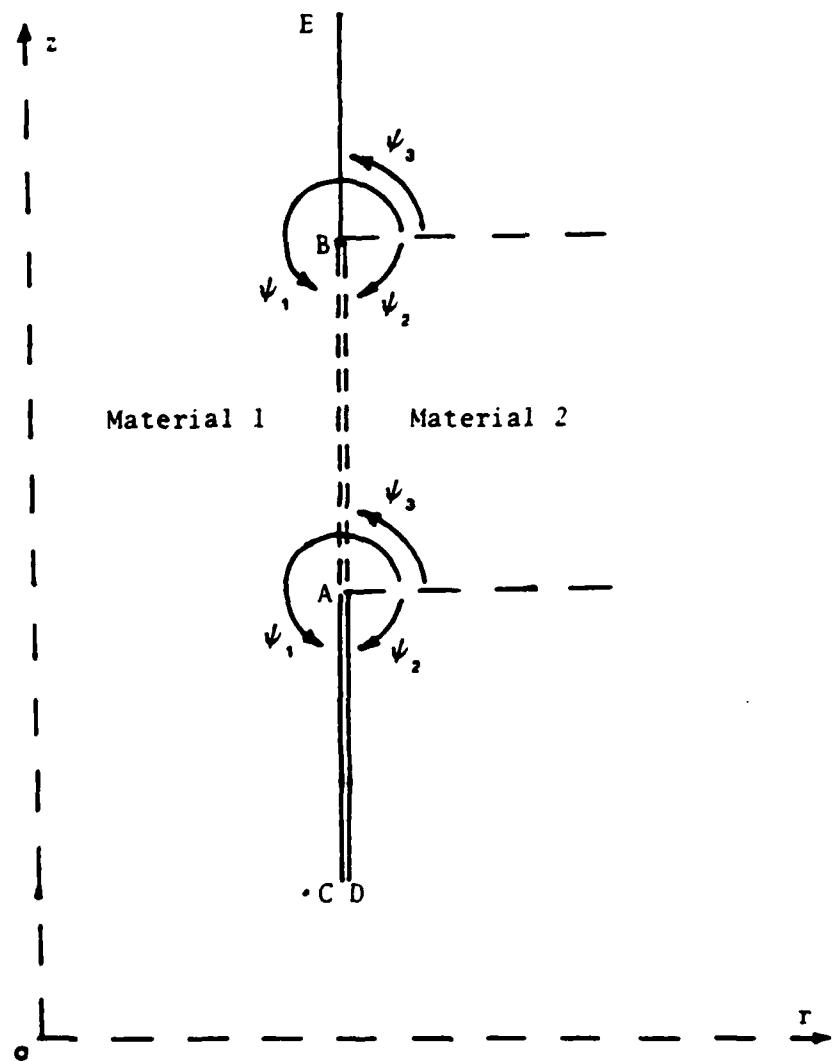


Fig.10 Cross section of an axisymmetric composite that contains a vertical interface crack with a contact zone

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EVALUATION OF A SIMILAR POLYMER
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STRUCTURES AND THEIR
APPLICATIONS TO STRESS ANALYSIS
OF A BORON FIBER

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Technical Report, MTL TR 87-1, January 1987, 69 pp.
Liver-Eals, Contract MA86-01-0007
D/A Project 585-0010
Final Report, 4/1/88 to 6/31/88

Now a transversely isotropic elastic body that contains a notch or a crack is under an antisymmetric deformation. It is shown that the displacement gradient near the singular point is in the form of a power series $\rho r^{(p,1)}(r, \theta)$... In which ρ is the polar coordinate with origin at the singular point and r is the displacement, or the order of singularity. A difficulty arises when ρ is as well as r when ρ is a positive integer or zero. In this case, the higher order terms of the series solution may not exist. A modified solution is required and is presented here. The modified solution has the new terms ρ^p in $\rho r^{(p,1)}(r, \theta)$... In $\rho r^{(p,1)}(r, \theta)$... as an application, we consider the interface near a broken fiber in a composite which is under an asymmetric deformation. The interface between the broken fiber and the matrix also suffers a delamination. This creates stress singularities at several points some of which require the modified configurations presented here.

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